Continued fractions of even type related to amphicheiral two bridge links

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Mathematical Science of Knots IV

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References

[1] T. Kadokami
The link-symmetric groups of 2-bridge links,
J. Knot Theory Ramif. **20** (8) (2011), 1129–1144.

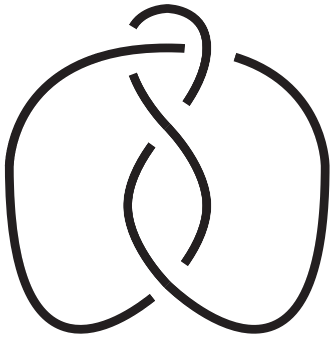
[2] Y. Jibiki
Talks in Hokuriku Number Theory Seminar
2021.3.4, 2021.11.18.

[3] C. Kraaikamp and A. Lopes
The theta group and the continued fraction expansion with even partial quotients, Geomeriae Dedicata **59** (1996), 293–333.

[4] I. Short and M. Walker
Even-integer continued fractions and the Faray tree,
Symmetries in Graphs, Maps, and Polytopes Workshop (2016), 13p.
arXiv: 1508.01373.

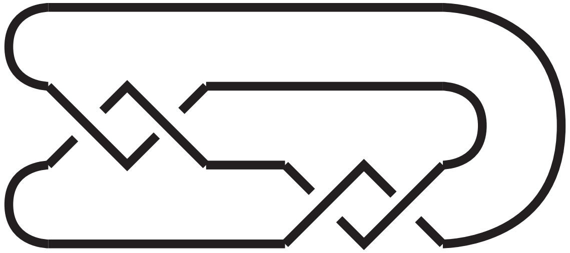
Contents

§1. Introduction

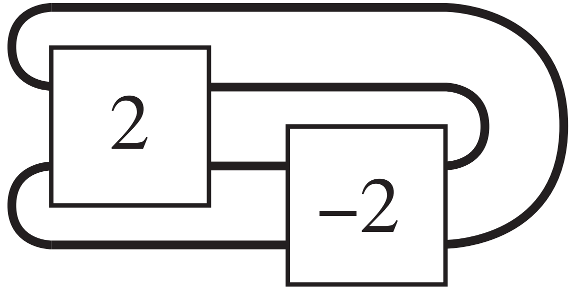


4_1

$\cong$



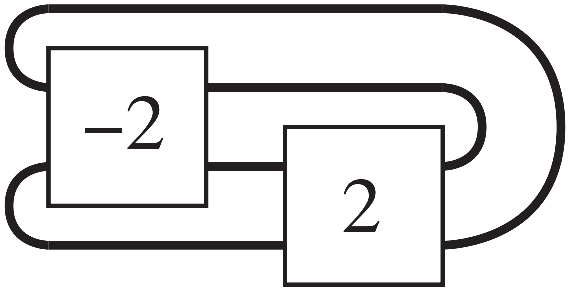
$\cong$



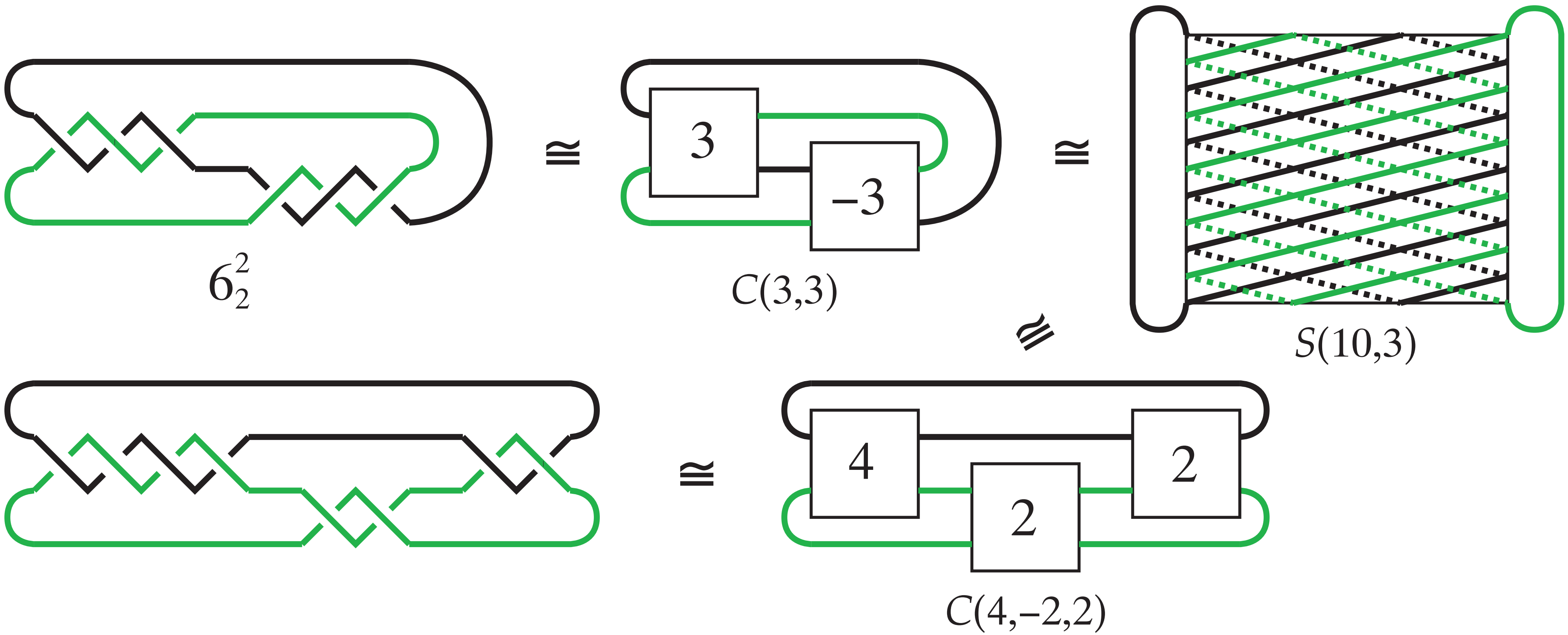
$C(2,2)$

$\ncong$

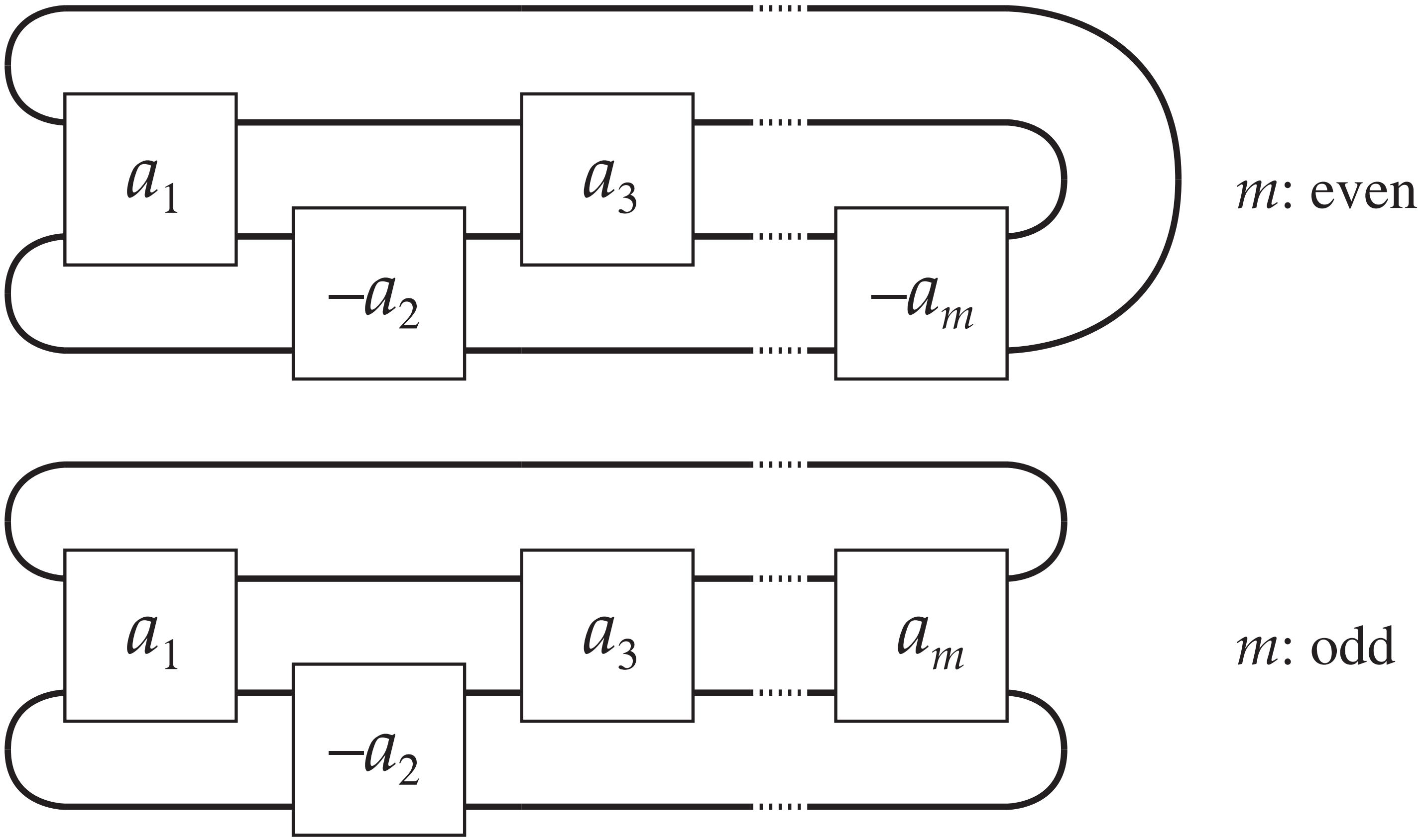
4_1^* $\cong$



$C(-2,-2)$



$$3 + \frac{1}{3} = \frac{10}{3} = 4 + \cfrac{1}{-2 + \cfrac{1}{2}}. \qquad 3^2 = 9 \equiv -1 \pmod{10}.$$



$C(a_1, a_2, \ldots, a_m)$

$$[a_1, \ldots, a_m] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \cfrac{\ddots}{a_{m-1} + \cfrac{1}{a_m}}}} = \frac{p}{q} = \frac{-p}{-q}$$

$$a_i, p, q \in \mathbb{Z} \setminus \{0\} \ (i = 1, \ldots, m), \ \gcd(p, q) = 1 \ (|p| \geq |q|).$$

Theorem 1.1 $C(a_1, \ldots, a_m) \cong S(p, q).$

$$(\star) \quad \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$$

Lemma 1.2

(1) $ps - qr = (-1)^m$.

(2) $qr \equiv (-1)^{m+1} \pmod p$.

(3) $[a_1, \ldots, a_{m-1}] = r/s$.

(4) $[a_m, \ldots, a_1] = p/r$.

Assumption $|a_i| = 1 \Longrightarrow a_{i-1}a_i > 0 \ \& \ a_i a_{i+1} > 0$.

Definition

$[a_1, \ldots, a_m]$, $(a_1, \ldots, a_m)$ $({}^\forall a_i \in \mathbb{Z} \setminus \{0\}; i = 1, \ldots, m)$: even type $\Longleftrightarrow$ ${}^\forall a_i$: even $(i = 1, \ldots, m)$.

Definition

$L = K_1 \cup \cdots \cup K_n$: n-component link

L^* : mirror image of L

L : amphicheiral $\Longleftrightarrow$ $L \cong L^*$ as unoriented links.

L : oriented, L : invertible $\Longleftrightarrow$ $L \cong -L$ as oriented links.

L : oriented ordered,

L : $(\varepsilon_1, \ldots, \varepsilon_n; \sigma)$-amphicheiral $\Longleftrightarrow$

$L^* = \varepsilon_{\sigma(1)} K_{\sigma(1)} \cup \cdots \cup \varepsilon_{\sigma(n)} K_{\sigma(n)}$ as oriented ordered knots

$(\varepsilon_i \in \{\pm\},\ \sigma \in \mathfrak{S}_n)$.

In particular, $\sigma = \iota \Longrightarrow$ we may omit σ.

Lemma 1.3

(1) A 2-bridge knot/link is strongly invertible. $S(p, q) \cong S(-p, -q)$.

(2) $S(p, q)^* \cong S(p, -q)$, $C(a_1, \ldots, a_m)^* \cong C(-a_1, \ldots, -a_m)$.

(3) A 2-component 2-bridge link is interchangeable.

§2. Basic properties

Lemma 2.1

(1) $[a_1, \ldots, a_m] = p/q$: even type

m : even

$\Longrightarrow p$: odd, q : even, r : even, s : odd, $\quad qr \equiv -1 \pmod{p}$,

$C(a_1, \ldots, a_m) = S(p, q)$: knot,

Seifert genus $g = m/2$.

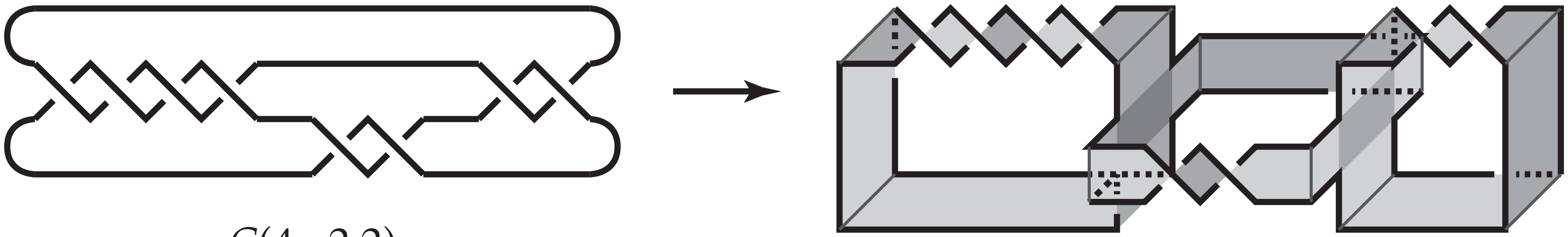
(2) $[a_1, \ldots, a_m] = p/q$: even type

m : odd

$\Longrightarrow p$: even, q : odd, r : odd, s : even, $\quad qr \equiv 1 \pmod{2p}$,

$C(a_1, \ldots, a_m) = S(p, q)$: 2-component link,

Seifert genus $g = (m-1)/2$.



C(4,–2,2)

Theorem 2.2

(1) $L(p,q)$: 2-fold branched covering over $S(p,q)$.

(2) p : odd, $S(p,q) \cong S(p',q')$
$\Longrightarrow\ p = p'$ & $q = q'$ or $qq' \equiv 1 \pmod{p}$.

(3) p : even, $S(p,q) \cong S(p',q')$ as oriented links
$\Longrightarrow\ p = p'$ & $q = q'$ or $qq' \equiv 1 \pmod{2p}$.

(4) p : even, $S(p,q) = K_1 \cup K_2$ as an oriented link,
$S(p',q') = K_1 \cup (-K_2) \Longrightarrow\ p = p'$ & $q' \equiv q + p \pmod{2p}$.

Corollary 2.3
$S(p,q)$: amphicheiral $\Longleftrightarrow q^2 \equiv -1 \pmod{p}$.

$q' = -q,\ q(-q) \equiv 1 \pmod{p}$. $\square$

Remark $q^2 \equiv -1 \pmod{p} \Longleftrightarrow s(q,p) = 0$.

§3. Main Theorems

Main Theorem 1

L : amphicheiral 2-bridge knot/link

$\iff {}^{\exists}m$: even &

${}^{\exists}a_i \in \mathbb{Z} \setminus \{0\}$ $(i = 1, \ldots, m)$

s.t. ${}^{\forall}a_i = a_{m+1-i}$ & $L \cong C(a_1, \ldots, a_m)$: symmetric form.

In particular, L : knot $\Longrightarrow$ We can take ${}^{\forall}a_i$: even.

Main Theorem 1′

$p, q \in \mathbb{Z} \setminus \{0\}$ s.t. p or q : even, $q^2 \equiv -1 \pmod p$.

$\iff {}^{\exists}m$: even &

${}^{\exists}a_i \in \mathbb{Z} \setminus \{0\}$ $(i = 1, \ldots, m)$

s.t. ${}^{\forall}a_i = a_{m+1-i}$ & $[a_1, \ldots, a_m] = p/q$.

Suppose m : odd. Then $m = 2g + 1$.
$\mathcal{E}_m = \{(a_1, \ldots, a_m)$: even type$\}$,
$\mathcal{E}_m^+ = \{(a_1, \ldots, a_m) \in \mathcal{E}_m \mid a_1 > 0\}$,
$\mathcal{E}_m^- = \{(a_1, \ldots, a_m) \in \mathcal{E}_m \mid a_1 < 0\}$,

$$\mathcal{E} = \bigcup_{g=0}^{\infty} \mathcal{E}_{2g+1},\ \mathcal{E}^{\pm} = \bigcup_{g=0}^{\infty} \mathcal{E}_{2g+1}^{\pm}.$$

$$\mathcal{A}_g^{\pm} \subset \mathcal{E}_{2g+1}^{\pm},\ \mathcal{A}_g = \mathcal{A}_g^+ \cup \mathcal{A}_g^- \subset \mathcal{E}_{2g+1},\ \mathcal{A} = \bigcup_{g=0}^{\infty} \mathcal{A}_g$$

are generating sets by (A1)–(A6).

(A1) : $\mathcal{A}_0^+ = \{(2)\},\ \mathcal{A}_0^- = \{(-2)\}$.

(A2) : $(a_1, \ldots, a_m) \in \mathcal{A}_g \longleftrightarrow (a_m, \ldots, a_1) \in \mathcal{A}_g$.

(A3) : $(a_1, \ldots, a_m) \in \mathcal{A}_g^+ \longleftrightarrow (-a_1, \ldots, -a_m) \in \mathcal{A}_g^-$.

(A4) : $(a_1, \ldots, a_m) \in \mathcal{A}_g^+ \longleftrightarrow (a_1 + 2, a_2, \ldots, a_m, -2, 2) \in \mathcal{A}_{g+1}^+$.

(A5) : $(a_1, \ldots, a_m) \in \mathcal{A}_g^+ \longleftrightarrow (2, a, a_1, \ldots, a_m, -a - 2, 2) \in \mathcal{A}_{g+2}^+$,
where $a \neq -2$.

(A6) : $(a_1, \ldots, a_m) \in \mathcal{A}_g^- \longleftrightarrow (2, a, a_1, \ldots, a_m, -a, 2) \in \mathcal{A}_{g+2}^+$.

$\mathbf{a} = (a_1, \ldots, a_m) \in \mathcal{E},\ C(\mathbf{a}) = C(a_1, \ldots, a_m)$.

Main Theorem 2

$\mathbf{a} \in \mathcal{E}$, $\quad C(\mathbf{a})$: amphicheiral $\Longleftrightarrow$ $\mathbf{a} \in \mathcal{A}$: even amphicheiral form.

Example

(1) $\mathbf{a}_g = (2g+2, \underbrace{-2, 2}_{1}, \ldots, \underbrace{-2, 2}_{g}) \in \mathcal{E}_g^+, \quad L_g = C(\mathbf{a}_g)$.

$$\mathbf{a}_0 = (2) \xrightarrow{\text{(A4)}} \mathbf{a}_1 = (4, -2, 2) \xrightarrow{\text{(A4)}} \cdots \xrightarrow{\text{(A4)}} \mathbf{a}_g \in \mathcal{A}_g^+.$$

$L_g = C(2g+1, 2g+1)$.

(2) $\mathcal{A}_1^+ = \{(4, -2, 2), (2, -2, 4)\}$,

$\mathcal{A}_2^+ = \{(6, -2, 2, -2, 2), (4, -2, 4, -2, 2), (2, -2, 2, -2, 6),$
$(2, -2, 4, -2, 4), (2, a, 2, -a-2, 2)\ (a \neq -2), (2, a, -2, -a, 2)\}$.

§4. Problems

Problem 1 What happens if we replace (A1) with another ?

Problem 2 About Main Theorem 2, is there more direct characterization ?

Problem 3 What is a translation between even amphicheiral form and symmetric form ?

Problem 4 Is there purely number theoretical proof of Main Theorem 2 ?

Problem 5 What are the corresponding notions of ♯ of components, linking number etc. in number theory ?

ご清聴ありがとうございました。