Chain-level MOY relations on Khovanov-Rozansky homology

結び目の数理 IV

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Abstract

Khovanov-Rozansky ホモロジーは、結び目の $\mathfrak{sl}_n$-不変量及び HOMFLY 多項式を圏化するものとして 2004 年に提案された。 これは、マトリックスファクトリゼーションのホモトピー論を用いて記述され、その Reidemeister 移動による不変性は MOY 関係式の圏化が本質である。 ところが、Khovanov-Rozansky のオリジナルの証明は、導来圏の同型射を用いて行われており、鎖複体のレベルでの具体的な記述はまだ知られていない。
上記の困難について、今回この MOY 関係式の圏化を鎖複体に対して直接構成する方法が得られたので、それについて解説する。 基本的なアイデアは、マトリックスファクトリゼーションを dg 加群の言葉で整理しなおし、代数トポロジーの手法を用いて KR ホモロジーを再構成することである。
本研究は、伊藤昇氏、中兼啓太氏との共同研究である。

Overview

WANT to realize knot operations on KR-homology in the **chain-level**:

- Reidemeister moves;
- Crossing changes;
- C_n-moves.

$\rightsquigarrow$ "higher dimensional" information of knots.

TODAY

Explicit chain homotopy equivalences that categorifies MOY relations of types I and II.

Background and Motivations

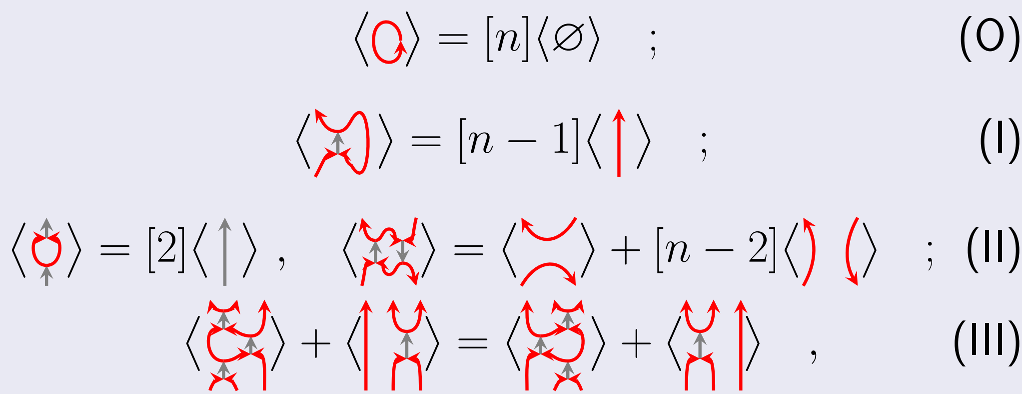
$\mathfrak{sl}_n$-polynomial and MOY calculus

Theorem 1.1 (Murakami, Ohtsuki, and Yamada [1998])

The representations of $\mathfrak{sl}_n$ yields a map

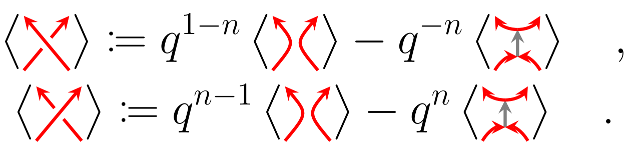
$$\{\textit{weighted } 3\textit{-valent graphs}\} \to \mathbb{Z}[\![q]\!] \; ; \quad G \mapsto \langle G \rangle$$

satisfying the following **MOY relations***:*



here $[i] := \frac{q^i - q^{-i}}{q - q^{-1}}$.

Extend $\langle - \rangle$ to knot diagrams by



$\rightsquigarrow$ **$\mathfrak{sl}_n$-polynomial** P_n is obtained.

Khovanov-Rozansky homology

Theorem 1.2 (Khovanov and Rozansky [2008a, 2008b]; Rasmussen [2015])

For each $N \geq 2$, there is a link homology $H_N(L) = \{H_N^{i,j}(L)\}_{i,j}$ such that

$$\sum_{i,j} (-1)^j q^i \dim_{\mathbb{Q}} H_N^{i,j}(L) \ \propto \ P_N(L) \quad ,$$

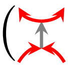
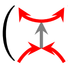
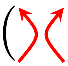
where $P_N(L)$ is the $\mathfrak{sl}_N$-polynomial of a link L.

$H_N(L)$ is called **Khovanov-Rozansky homology**;

$\rightsquigarrow$ **categorification** of $\mathfrak{sl}_N$-polynomial.

Sketch of the construction

- Trivalent graph $G \mapsto$ **matrix factorization** $C_p(G)$;
  $\rightsquigarrow$ an object of the **dg-category** MF.
- MOY relations in the **derived category** $D(\mathrm{MF})$.
- Extend $C_p(G)$ to link diagrams by giving

C_p () $\to C_p$ () , C_p () $\to C_p$ () ,

  and considering the **complexes in** $D(\mathrm{MF})$.
- Observe that MOY $\Rightarrow$ Reidemeister moves.

Motivating question

Question

Can we realize **MOY relations** and **Reidemeister moves** in $\mathbf{MF}$ rather than $D(\mathbf{MF})$.

WHY?

- Derived categories are difficult.
- In fact, $\mathbf{MF}$ has strictly richer **homotopy theory** than $D(\mathbf{MF})$ does.
  $\rightsquigarrow$ e.g. expect of **(co)homology operations**.
- **Vassiliev theory** on KR homology.
  (**higher commutativity** of Reidemeister moves)
  $\rightsquigarrow$ Realization of **"wall-crossing"** of [Shirokova & Webster, 2007].

Strategy

MOY 0,I relations are already realized in $\mathbf{MF}$.

$\rightsquigarrow$ Realize MOY II,III in $\mathbf{MF}$ rather than $D(\mathbf{MF})$.

Idea

Take a **projective resolution** of $C_p(G)$ as a matrix factorization.

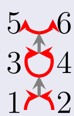
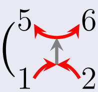
Main result

For a graph G, we define $\widetilde{C}_p(G)$ to be a complex generated by elements of the form $\gamma_n\omega$, $\bar{\theta}\gamma_n\omega$ where

- $n \in \mathbb{N}$; and
- $\omega \in \bigwedge V$ with V associated to each gray edges and bivalent vertices in G.

Main Theorem (Ito-Nakagane-Y.)

There is the following homotopy equivalences in $\mathbf{MF}$*:*

$\widetilde{C}_p\Big($  $\Big) \rightleftarrows \widetilde{C}_p\Big($  $\Big)^{\oplus 2}$

$$q(X_3,X_4)\gamma_n v_I^{\perp} \longmapsto \begin{bmatrix} \frac{q(X_1,X_2)-q(X_2,X_1)}{X_1-X_2} \\ \frac{X_1 q(X_2,X_1)-X_2 q(X_1,X_2)}{X_1-X_2} \end{bmatrix} \gamma_n v_I$$

$$q(X_3,X_4)\gamma_n v_I^{\top} \longmapsto 0$$

$$(a+bX_3)(\gamma_n v_I + \gamma_{n-1}(\cdots) + \gamma_{n-2}(\cdots)) \longleftarrow \begin{bmatrix} a \\ b \end{bmatrix} \gamma_n v_I$$

Remark

In terms of the matrix factorization $\widetilde{C}_p(G)$, we have already seen that MOY III can be realized in $\mathbf{MF}$.

$\rightsquigarrow$ Explicit computation is now in progress.

Khovanov-Rozansky complex

Matrix factorizations

R: a fixed coefficient ring.

Definition

A **matrix factorization with potential** $w \in R$ is a graded R-module $M = M^\bullet$ with two differentials

$$d_+ : M^\bullet \to M^{\bullet+1}, \quad d_- : M^\bullet \to M^{\bullet-1} \quad ,$$

such that $d_+d_- + d_-d_+ = w \cdot \mathrm{id} : M^\bullet \to M^\bullet$.

$\rightsquigarrow$ $\mathbf{MF}_w = \mathbf{MF}_w(R)$: the category of matrix factorizations with potential w.

Example (Koszul factorization)

V: a free R-module with basis $\{v_1, \dots, v_r\}$;

$\vec{a} := (a_1, \dots, a_n), \vec{b} = (b_1, \dots, b_n) \in R^{\times n}$

$\rightsquigarrow$ The matrix factorization $(\vec{a} \;\; \vec{b})$ with potential $\vec{a} \cdot \vec{b}$:

- as a graded module, $(\vec{a} \;\; \vec{b})^n \cong V^{\wedge - n}$;
- the differential d_+ is defined by $d_+(v_i) = a_i$ and the Leibniz rule for the exterior product $\wedge$;
- the differential d_- is given by

$$d_-(\omega) := \omega \wedge \left(\textstyle\sum_{i=1}^r a_i v_i\right) \quad .$$

Khovanov-Rozansky complex

Definition

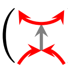
A **KR graph** is an oriented planar graph G with two types of edges and with vertices of the form

Definition

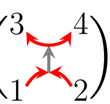
G: KR graph $\rightsquigarrow R(G) := \mathbb{K}[X_e : e \in E^{赤}(G)]$.

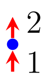
Ingredients G: KR graph, $p(X) \in \mathbb{Z}[X]$

Construction $C_p(G) \in \mathbf{MF}_{w(G)}$ is given by

$C_p(G) := \Big(\bigotimes_{\text{gray edges in } G} C_p\big($  $\big)\Big) \otimes \Big(\bigotimes_{\text{bivalent vertices}} C_p\big($  $\big)\Big)$,

with

$C_p\Big($  $\Big) := \begin{pmatrix} \theta^{(1)} & X_3 + X_4 - X_1 - X_2 \\ \theta^{(2)} & X_3X_4 - X_1X_2 \end{pmatrix}$,

$C_p\Big($  $\Big) := \begin{pmatrix} \frac{p(X_2)-p(X_1)}{X_2-X_1} & X_2 - X_1 \end{pmatrix}$,

where $\theta^{(1)}, \theta^{(2)} \in R(G)$ is taken so that

$$(X_3 + X_4 - X_1 - X_2)\theta^{(1)} + (X_3X_4 - X_1X_2)\theta^{(2)} = p(X_3) + p(X_4) - p(X_1) - p(X_2) \quad .$$

Khovanov-Rozansky complex

$w(G) \coloneqq \sum_{e \in \partial_{\text{out}} G} p(X_e) - \sum_{e \in \partial_{\text{in}} G} p(X_e)$

Fact

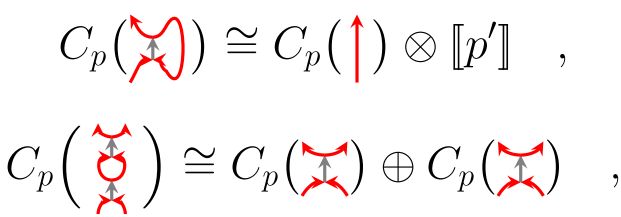
$C_p(G)$ is a **matrix factorization** with potential $w(G)$.

$\rightsquigarrow C_p(G) \in \mathbf{MF}_{w(G)}(R(G)) \xrightarrow{\textit{forget}} \mathbf{MF}_{w(G)}(R(\partial G))$.

Theorem 2.1 (Khovanov and Rozansky [2008a]; Rasmussen [2015])

$C_p(-)$ satisfies **MOY relations** *in $D(\mathbf{MF}_w(R(\partial G)))$.*

Especially, in $D(\mathbf{MF}_w)$, we have



where $[\![p']\!]$ is a complex that categorifies $[n-1]$ when $p(X) = X^n$.

Remark

This does not imply these are homotopy equivalent in $\mathbf{MF}_w$.

Non-invertibility of reduction

In MOY relations in $D(\mathbf{MF}_w)$, the key is

Lemma 2.2

R: commutative ring

$a(X) \in R[X]$, $b \in R$, $\vec{c}(X), \vec{d}(X) \in R[X]^{\times n}$ *such that*

$$w := a(X)(X-b) + \vec{c}(X) \cdot \vec{d}(X) \in R \quad .$$

Then, there is a quasi-isomorphism in $\mathbf{MF}_w(R)$:

$$\rho : \begin{pmatrix} a(X) & X-b \\ \vec{c}(X) & \vec{d}(X) \end{pmatrix} \xrightarrow{\simeq} \begin{pmatrix} \vec{c}(b) & \vec{d}(b) \end{pmatrix} \quad .$$

The source of the problem

ρ does not have a homotopy inverse.

Remark

[Rasmussen, 2015] constructed a homotopy inverse of ρ as a chain map while it is not a morphism of matrix factorizations.

Main result

Projective resolution of matrix factorizations

The dg algebra $A_w(R)$

- $A_w(R) := R[\theta]/(\theta^2)$ with $|\theta| = -1$ as a graded alg.;
- the differential is given by $d\theta = w$.

Lemma 3.1

$\mathbf{MF}_w \cong \mathbf{dgMod}(A_w(R))$.

The dg algebra $\widetilde{A}_w(R)$

- $\widetilde{A}_w(R) := R\langle \theta, \bar{\theta}, \gamma_n : n \in \mathbb{N} \rangle$ with

$$|\theta| = |\bar{\theta}| = -1 , \quad |\gamma_n| = -2n \quad ,$$

$$\theta^2 = \bar{\theta}^2 = 0 , \quad \theta\bar{\theta} = -\bar{\theta}\theta , \quad \theta\gamma_n = \gamma_n\theta , \quad \bar{\theta}\gamma_n = \gamma_n\bar{\theta} \quad ,$$

$$\gamma_m\gamma_n = \binom{m+n}{m}\gamma_{m+n} \quad ;$$

- the differential is given by

$$d(\theta) = d(\bar{\theta}) = w , \quad d(\gamma_n) = (\bar{\theta} - \theta)\gamma_{n-1} \quad ,$$

where $\gamma_{-1} = 0$

Proposition 3.2

If $M \in \mathbf{MF}_w(R)$ is bounded and degree-wise projective over R, then $\widetilde{P}(M) := \widetilde{A}_w(R) \otimes_{A_w(R)} M$ is a **projective resolution** *of M.*

Homotopy inverse to reduction

Proposition 3.3

R: commutative ring

$a(X) \in R[X]$, $b \in R$, $\vec{c}(X), \vec{d}(X) \in R[X]^{\times n}$ *such that*

$$w \coloneqq a(X)(X-b) + \vec{c}(X) \cdot \vec{d}(X) \in R \quad .$$

Then, there is a homotopy equivalence in $\mathbf{MF}_w(R)$

$$\widetilde{P}(\rho) : \widetilde{P}\begin{pmatrix} a(X) & X-b \\ \vec{c}(X) & \vec{d}(X) \end{pmatrix} \leftrightarrows \widetilde{P}\begin{pmatrix} \vec{c}(b) & \vec{d}(b) \end{pmatrix} : \widetilde{\iota}$$

given by

$$\widetilde{P}(\rho)(f(X)\gamma_n v_0) = 0 \,, \quad \widetilde{P}(\rho)(f(X)\gamma_n v_i) = f(b)v_i \quad ,$$

$$\widetilde{\iota}(\gamma_n v_I) = \gamma_n\big(v_I - \sum_{i \in I} \mathrm{sgn}(i, I \setminus i)\frac{d_i(X) - d_i(b)}{X-b} v_0 v_{I \setminus i}\big) - \gamma_{n-1}\big(\sum_i \frac{c_i(X) - c_i(b)}{X-b} v_0 v_i v_I\big) \quad ,$$

where

$$v_I = v_{i_1} \ldots v_{i_k} \quad .$$

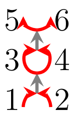
for $I = \{i_1 < \cdots < i_k\} \subset \{1, \ldots, n\}$.

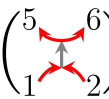
Construction of the homotopy equivalences

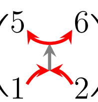
Set $R \coloneqq \mathbb{K}[X_1, X_2, X_5, X_6]$ and

$$\overline{R} \coloneqq R[X_3, X_4]\ , \quad S \coloneqq \overline{R}^{\mathfrak{S}_2} \cong R[s_1, s_2] \quad .$$

Sequence of homotopy equivalences

$\widetilde{C}_p \Bigg($  $\Bigg) \cong \overline{R} \otimes_S \begin{pmatrix} \theta^{(1)}_{\text{上}} & X_5 + X_6 - s_1 \\ \theta^{(2)}_{\text{上}} & X_5 X_6 - s_2 \\ \theta^{(1)}_{\text{下}} & s_1 - X_1 - X_2 \\ \theta^{(2)}_{\text{下}} & s_2 - X_1 X_2 \end{pmatrix}$

$\xrightarrow{\text{reduction}} \dfrac{\overline{R}}{(X_3 + X_4 - X_1 - X_2,\ X_3 X_4 - X_1 X_2)} \otimes_R \widetilde{C}_p \Big($  $\Big)$

$\cong \widetilde{C}_p \Big($  $\Big)^{\oplus 2}$

$\rightsquigarrow$ An explicit inverse to "reduction" (Proposition 3.3).

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