Quandle twisted Alexander invariants and homology groups

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Twisted Alex. inv. (Wada)

K: an ori. knot in S^3, $G(K) = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$: the knot group, R: a UFD.
$\alpha : G(K) \to \mathbb{Z}$: the abelianization, $\rho : G(K) \to GL(k; R)$: a group hom.

$$\rightsquigarrow \left((\rho \otimes \alpha) \circ \mathrm{pr} \left(\frac{\partial r_i}{\partial x_j} \right) \right) \in M(m, n; M(k, k; R[t^{\pm 1}]))$$

$$\rightsquigarrow \Delta_{K,\rho}(t) := \frac{\Delta \left(\left((\rho \otimes \alpha) \circ \mathrm{pr} \left(\frac{\partial r_i}{\partial x_j} \right) \right)_i \right)}{\det((\rho \otimes \alpha) \circ \mathrm{pr}(x_i - 1))}$$
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Twisted Alex. inv. for quandles (Ishii-Oshiro)

$Q = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$: a quandle, X: a quandle, R: a UFD, $c \in X$.
$\rho : Q \to X$: a quandle hom. $\boldsymbol{f} = (f_1, f_2)$: an **Alexander pair** of maps
$f_1, f_2 : X^2 \to M(k, k; R)$.
$\rightsquigarrow A(Q, \rho; f_1, f_2) \in M(m, n; M(k, k; R))$: an $\boldsymbol{f}$-twisted Alexander matrix.
$\rightsquigarrow \Delta(A(Q, \rho; f_1, f_2), c) := \dfrac{\Delta(A(Q, \rho; f_1, f_2)_i)}{\det(f_2(\rho(x_i) \bar{*} c, c))}$.

Theorem (Kirk-Livingston, 1999)

Assume that R is a field.
$V = R[t^{\pm 1}]^k$, $E(K) = S^3 \backslash \mathrm{int}(N(K))$: *the exterior of K.*
Then, $\Delta_{K,\rho}(t) \doteq \dfrac{\mathrm{order}(H_1(E(K); V_\rho))}{\mathrm{order}(H_0(E(K); V_\rho))}$.

Goal.
We interpret $\Delta(A(Q, \rho; f_1, f_2), c)$ in terms of the homology group.
We focus on the homology group introduced by Andruskiewitsch-Graña.

Quandle

Definition (Joyce, Matveev, 1982)

X: a set, $* : X^2 \to X$: a binary operation.
$X = (X, *)$: **quandle**
$\Leftrightarrow$ $*$ satisfies the following conditions:

1. ${}^{\forall}x \in X$, $x * x = x$.
2. ${}^{\exists}\bar{*} : X^2 \to X$ s.t. ${}^{\forall}x, y \in X, (x * y)\bar{*}y = (x\bar{*}y) * y = x$.
3. ${}^{\forall}x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

Ex.
G: a group, $x * y := y^{-1}xy$ $(x, y \in G)$.
$\mathrm{Conj}(G) = (G, *)$: the **conjugation quandle of** G.

Alexander pair

Definition (Ishii-Oshiro (cf. Andruskiewitsch-Graña, 2003))

X: a quandle, R: a ring, $f_1, f_2 : X^2 \to R$: maps.
$\boldsymbol{f} = (f_1, f_2)$: an **Alexander pair**
$\Leftrightarrow$ f_1 and f_2 satisfy the following conditions:

- ${}^\forall x \in X$, $f_1(x,x) + f_2(x,x) = 1$.
- ${}^\forall x, y \in X$, $f_1(x,y)$ is a unit of R.
- ${}^\forall x, y, z \in X$,

$$f_1(x*y,z)f_1(x,y) = f_1(x*z,y*z)f_1(x,z),$$
$$f_1(x*y,z)f_2(x,y) = f_2(x*z,y*z)f_1(y,z) \text{ and}$$
$$f_2(x*y,z) = f_1(x*z,y*z)f_2(x,z) + f_2(x*z,y*z)f_2(y,z).$$

Ex.

(1) X: a quandle, $f_1, f_2 : X^2 \to \mathbb{Z}[t^{\pm 1}]$: maps defined by

$$f_1(x, y) := t^{-1}, \quad f_2(x, y) := 1 - t^{-1}.$$

Then, (f_1, f_2): an Alexander pair.

(2) G: a group, $X = \mathrm{Conj}(G)$, R: a commutative ring.

$f_1, f_2 : X^2 \to R[G][t^{\pm 1}]$: maps defined by

$$f_1(x, y) := y^{-1}t^{-1}, \quad f_2(x, y) := y^{-1}x - y^{-1}t^{-1}.$$

Then, (f_1, f_2): an Alexander pair.

$Q = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle, X$: quandles, $\rho : Q \to X$ a quandle homomorphism.
R: a UFD, $\boldsymbol{f} = (f_1, f_2)$: an Alexander pair of maps $f_1, f_2 : X^2 \to M(k, k; R)$.

$\rightsquigarrow A(Q, \rho; f_1, f_2) = \left(\frac{\partial_{\boldsymbol{f} \circ \rho}}{\partial x_j}(r_i) \right) \in M(m, n; M(k, k; R))$
: the $\boldsymbol{f}$-twisted Alexander matrix w.r.t. $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$.

$A(Q, \rho; f_1, f_2)_i \in M(m, n-1; M(k, k; R))$
:the matrix obtained from $A(Q, \rho; f_1, f_2)$ by removing the i-th column.

We regard $A(Q,\rho;f_1,f_2)_i$ as a $km \times k(n-1)$ matrix over R.
$\Delta(A(Q,\rho;f_1,f_2)_i)$: the gcd of all $k(n-1)$-minors of $A(Q,\rho;f_1,f_2)_i$.
$c \in X$: fixed

Proposition (Ishii-Oshiro)

${}^{\forall} i, j$, $\det(f_2(\rho(x_j)\bar{*}c, c))\Delta(A(Q,\rho;f_1,f_2)_i) \doteq \det(f_2(\rho(x_i)\bar{*}c, c))\Delta(A(Q,\rho;f_1,f_2)_j)$.

Assume that ${}^{\exists} i$ s.t. $\det(f_2(x_i\bar{*}c, c)) \neq 0$.

$\Delta(A(Q,\rho;f_1,f_2),c) := \dfrac{\Delta(A(Q,\rho;f_1,f_2)_i)}{\det(f_2(\rho(x_i)\bar{*}c, c))}$ does not depend on the choice of i.

Theorem (Ishii-Oshiro)

Q': a quandle equipped with a finite presentation.
If ${}^{\exists}\psi : Q' \to Q$: a quandle isomorphism,
then $\Delta(A(Q,\rho;f_1,f_2),c) \doteq \Delta(A(Q',\rho\circ\psi;f_1,f_2),c)$.

K: an oriented knot in S^3.
(1) X: a quandle, $f_1, f_2 : X^2 \to \mathbb{Z}[t^{\pm 1}]$: maps defined by

$$f_1(x, y) := t^{-1}, \quad f_2(x, y) := 1 - t^{-1}.$$

$\Rightarrow \Delta(A(Q(K), \rho; f_1, f_2), c) \doteq \dfrac{\Delta_K(t)}{t-1}.$
(2) R: a UFD, $G = GL(k; R)$, $X = \mathrm{Conj}(G)$, $\rho : Q(K) \to X$: a quandle hom.
$f_1, f_2 : X^2 \to M(k, k; R[t^{\pm 1}])$: maps defined by

$$f_1(x, y) := y^{-1}t^{-1}, \quad f_2(x, y) := y^{-1}x - y^{-1}t^{-1}.$$

$\rho_{\mathrm{grp}} : G(K) \to G$: the induced group homomorphism by $\rho : Q(K) \to X$.
$\Rightarrow \Delta(A(Q(K), \rho; f_1, f_2), c) \doteq \Delta_{K, \rho_{\mathrm{grp}}}(t)$.

$$[q_1, q_2, \ldots, q_n] = ((q_1 * q_2) * \cdots) * q_n.$$

Definition (Andruskiewitsch-Graña, 2003)

For each $n \in \mathbb{Z}_{>0}$, $C_n^{\boldsymbol{f}\circ\rho}(Q)$: the free $M(k,k;R)$-module whose basis is Q^n.
$C_0^{\boldsymbol{f}\circ\rho}(Q) := M(k,k;R)$.
For each $n \in \mathbb{Z}_{>1}$, $\partial_n^{\boldsymbol{f}\circ\rho} : C_n^{\boldsymbol{f}\circ\rho}(Q) \to C_{n-1}^{\boldsymbol{f}\circ\rho}(Q)$ defined by

$$\begin{aligned}
&\partial_n^{\boldsymbol{f}\circ\rho}(q_1, \ldots, q_n) \\
&= \sum_{i=2}^{n} (-1)^i f_1(\rho([q_1, \ldots, \hat{q}_i, \ldots, q_n]), \rho([q_i, \ldots, q_n]))(q_1, \ldots, \hat{q}_i, \ldots, q_n) \\
&- \sum_{i=2}^{n} (-1)^i (q_1 * q_i, \ldots, q_{i-1} * q_i, q_{i+1}, \ldots, q_n) \\
&+ f_2(\rho([q_1, q_3, \ldots, q_n]), \rho([q_2, \ldots, q_n]))(q_2, \ldots, q_n).
\end{aligned}$$

Definition (Andruskiewitsch-Graña, 2003)

$\partial_1^{\boldsymbol{f}\circ\rho} : C_1^{\boldsymbol{f}\circ\rho}(Q) \to C_0^{\boldsymbol{f}\circ\rho}(Q)$: defined by $\partial_1^{\boldsymbol{f}\circ\rho}(q) = f_2(\rho(q)\bar{*}c, c)$.
$C_\bullet^{\boldsymbol{f}\circ\rho}(Q) = (C_n^{\boldsymbol{f}\circ\rho}(Q), \partial_n^{\boldsymbol{f}\circ\rho})$: a chain complex.
$H_n^{\boldsymbol{f}\circ\rho}(Q)$: the n-th homology group of the chain complex $C_\bullet^{\boldsymbol{f}\circ\rho}(Q)$.

V: a right $M(k, k; R)$-module.
$C_\bullet^{\boldsymbol{f}\circ\rho}(Q; V) := (V \otimes C_n^{\boldsymbol{f}\circ\rho}(Q), \mathrm{Id}_V \otimes \partial_n^{\boldsymbol{f}\circ\rho})$.
$H_n^{\boldsymbol{f}\circ\rho}(Q; V)$: the n-th homology group of the chain complex $C_\bullet^{\boldsymbol{f}\circ\rho}(Q; V)$

Ex.
$\partial_1^{\boldsymbol{f}\circ\rho}(x, y) = f_1(x, y)x - x * y + f_2(x, y)y$.

Remark.
If $c \in \mathrm{Im}(\rho)$, $C_\bullet^{\boldsymbol{f}\circ\rho}(Q)$ was introduced by Andruskiewitch-Grana.

Theorem A

Let us consider the complex: $M(k,k;R)^m \xrightarrow{\partial_2'} M(k,k;R)^n \xrightarrow{\partial_1'} M(k,k;R) \to 0$, *where*

$$\partial_2'(\boldsymbol{a}) = \boldsymbol{a}A(Q,\rho;f_1,f_2), \quad \partial_1'(\boldsymbol{a}) = \boldsymbol{a}\begin{pmatrix} f_2(\rho(x_1)\bar{*}c, c) \\ \vdots \\ f_2(\rho(x_n)\bar{*}c, c) \end{pmatrix}.$$

Then, $H_1^{\boldsymbol{f}\circ\rho}(Q) \cong \mathrm{Ker}(\partial_1')/\mathrm{Im}(\partial_2')$ *and* $H_0^{\boldsymbol{f}\circ\rho}(Q) \cong M(k,k;R)/\mathrm{Im}(\partial_1')$.

Assume that R is a PID.

$V = R^k$: an $(R, M(k,k;R))$-module.

$$V \otimes M(k,k;R)^m \xrightarrow{\mathrm{Id}_V \otimes \partial_2'} V \otimes M(k,k;R)^n \xrightarrow{\mathrm{Id}_V \otimes \partial_1'} V \otimes M(k,k;R) \to 0.$$

$$\Rightarrow V^m = R^{km} \xrightarrow{\mathrm{Id}_V \otimes \partial_2'} V^n = R^{kn} \xrightarrow{\mathrm{Id}_V \otimes \partial_1'} V = R^k \to 0.$$

Hence, we can define the **order** of $H_n^{\boldsymbol{f} \circ \rho}(Q;V)$, which is denoted by $\mathrm{order}(H_n^{\boldsymbol{f} \circ \rho}(Q;V))$.

Theorem B

If $H_0^{\boldsymbol{f} \circ \rho}(Q;V)$ does not have a free part,
then we can define $\Delta(A(Q,\rho;f_1,f_2),c)$ and it holds that

$$\Delta(A(Q,\rho;f_1,f_2),c) \doteq \frac{\mathrm{order}(H_1^{\boldsymbol{f} \circ \rho}(Q;V))}{\mathrm{order}(H_0^{\boldsymbol{f} \circ \rho}(Q;V))}.$$

Remark.

Ishii and Oshiro defined $\Delta(A(Q,\rho;f_1,f_2),R(Q,\rho;\boldsymbol{f}_{\mathrm{col}}))$ using **column relation maps** $\boldsymbol{f}_{\mathrm{col}}=(f_{\mathrm{col},1},\ldots,f_{\mathrm{col},l})$.

$\Delta(A(Q,\rho;f_1,f_2),c)$ is a special case of $\Delta(A(Q,\rho;f_1,f_2),R(Q,\rho;\boldsymbol{f}_{\mathrm{col}}))$.

In the same way, we can also interpret $\Delta(A(Q,\rho;f_1,f_2),R(Q,\rho;\boldsymbol{f}_{\mathrm{col}}))$ in terms of the homology group.

Thank you for your attention.