同変特異インスタントン・フレアー理論による結び目不変量
Knot invariants from the equivariant singular instanton Floer theory

(partly based on a joint work with Aliakbar Daemi, Kouki Sato, Christopher Scaduto, and Masaki Taniguchi)

Hayato Imori (Kyoto University)

December 25, 2021

Backgrounds

- [Kronheimer-Mrowka 1993, 1995] solved Milnor conjecture:

$$g_{slice}(T_{p,q}) = \frac{(p-1)(q-1)}{2}$$

  by the technique of Yang-Mills gauge theory over $X \setminus \Sigma$.
- Floer theory from singular instanton are developed in [Collin-Steer 1999], [Kronheimer-Mrowka 2011] etc. $\to$ Knot homology groups are defined.
- [Daemi-Scaduto 2019] developed "local equivalence theory" for knots from equivariant singular Floer theory.
  $\longrightarrow$ $\mathcal{S}$-complex $\widetilde{C}(K)$, knot Frøyshov invariant $h(K) \in \mathbb{Z}$, Γ-invariant.
  $\longrightarrow$ Application to the 4D clasp number problem [Daemi-Scaduto 2020].

Some versions of singular instanton Floer homology can be interpreted as the "categorification of knot signature $\sigma(K)$ (or more generally Tristram-Levine signature)".

Developments of equivariant singular instanton Floer theory

Recently, Daemi-Scaduto's technique has developed further to solve problems on knot theory which cannot be reached from other Floer theory (Heegaard knot Floer, Khovanov homology, Seiberg-Witten theory)

Generalization [Imori 2021]

Daemi-Scaduto's construction extends to family of knot invariants followed by [Echeverria 2019].

$\longrightarrow$ Familly of $\mathcal{S}$-complex $\widetilde{C}^{\alpha}_{*}(K)$ with parameter $\alpha \in \mathbb{Q} \cap (0, \frac{1}{2})$ s.t. $\Delta_K(e^{4\pi i\alpha}) \neq 0$.

$\longrightarrow$Application to representation of knot group (Main result 1)

Refinement [Daemi-Imori-Sato-Scaduto-Taniguchi 2021]

A refinement of knot invariants can be defined using the filtered structure of S complex.

$\longrightarrow$ the r_s-invariant, $\mathcal{J}$-invariant for knots.

$\longrightarrow$Application to linear independence problems in the knot concordance groups (Main result 2 and 3).

- Yang-Mills gauge theory is strongly related to $SU(2)$-representations of fundamental groups. For example, the following is the one of the applications.

Theorem (Kronheimer-Mrowka 2004)

If $K \subset S^3$ is non-trivial knot then $\pi_1(S^3 \setminus K)$ admits non-abelian $SU(2)$-representations.

- How about problems related to concordance?

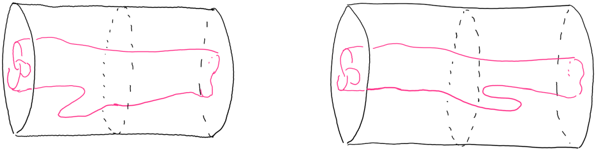
Definition

Two oriented knots K_0 and K_1 are concordant if there is a properly smoothly embedded annulus $S \subset [0,1] \times S^3$ such that $K_0 = S \cap \{0\} \times S^3$ and $K_1 = S \cap \{1\} \times S^3$.

$S : K_0 \to K_1$ denotes a concordance from K_0 to K_1.

Definition

A concordance $S : K_0 \to K_1$ is called ribbon if the projection $S^3 \times [0, 1] \supset S \to [0, 1]$ is a Morse function without local maximum.



Ribbon concordance　　　　Not ribbon concordance

When concordance is ribbon?
The following conjecture (the Slice-Ribbon conjecture) related to that question.

Conjecture (Fox 1962)

For any concordance $U \to K$, there exists a ribbon concordance $R : U \to K$.

On the other hand, [Daemi-Scaduto 2020] asked the following.

Question

Let $T_{p,q}$ be a (p, q)-torus knot. For any concordance $T_{p,q} \to K$, is there a ribbon concordance $T_{p,q} \to K$?

The evidence is given by

Theorem (Daemi-Scaduto 2020)

For a given concordance $S : T_{p,q} \to K$, any traceless $SU(2)$-representation of $\pi_1(S^3 \setminus T_{p,q})$ extends over the concordance complement.

"traceless" means that the homotopy class of the meridian is mapped to an traceless element in $SU(2)$.
In fact, it is known that

Proposition (Gordon 1981)

Let $R : K_1 \to K_2$ be a ribbon concordance, then any $SU(2)$-representation of $\pi_1(S^3 \setminus K_1)$ extends to the concordance complement.

Since the "traceless" condition in Daemi-Scaduto's theorem arises from technical reason, it is natural to ask the following.

Question

Con we drop the traceless condition in Daemi-Scaduto's theorem?

- The traceless condition can be dropped by generalizing Daemi-Scaduto's technique into family.

Theorem (Imori 2021)

For any concordance $S : T_{p,q} \to K$, any $SU(2)$-representation of $\pi_1(S^3 \setminus T_{p,q})$ extends to the concordance complement.

This result gives further evidence to the version of the Slice-Ribbon conjecture by [Daemi-Scaduto 2020].

- We can ask the further question.

Question

Let T be a iterated torus knot, and consider arbitrary concordance $S : T \to K$. Does any $SU(2)$-representations of $\pi_1(S^3 \setminus T)$ extend to the concordance complement?

This reduce to the following.

For iterated torus knot T, $|\mathcal{R}_\alpha(S^3 \setminus T, SU(2))| = \pm\frac{1}{2}\sigma_\alpha(T)$?

- Smooth concordance group

$$\mathcal{C} = \{K \subset S^3\}/\sim_{\text{concordance}}$$

We simply call $\mathcal{C}$ the concordance group.

- Algebraic concordance

$$\mathcal{C}_{\text{alg}} = \{\text{Seifert forms}\}/\sim_{\substack{\text{algebraic}\\\text{concordance}}}$$

- The algebraic concordance group has the structure $\mathcal{C}_{\text{alg}} \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$, and there is a surjective homomorphism $\mathcal{C} \to \mathcal{C}_{\text{alg}}$.
- More deeper structures of $\mathcal{C}$ has been studied. For example,
  The subgroup of topologically slice knots $\mathcal{C}_{\text{TS}} \subset \mathcal{C}$ has $\mathbb{Z}^\infty$-subgroup [Endo 1995].
  There exists a $\mathbb{Z}^\infty$-summand in $\mathcal{C}_{\text{TS}}$ [Hom 2015], [Ozsváth-Stipscitz-Szabó 2017].
  $\longleftarrow$To study the structures of $\mathcal{C}$, concordance invariants from Floer theory often provide powerful tools.

Knot invariants form Heegaard Floer theory, Khovanov honology, and Seiberg-Witten theory are "weak" for alternating knots.

- τ-invariant The concordance invariant $\tau : \mathcal{C} \to \mathbb{Z}$ was introduced by [Ozsváth-Szabó 2003].

Theorem (Ozsváth-Szabó 2003)

For alternating knot $K \subset S^3$, $\tau(K) = -\frac{1}{2}\sigma(K)$.

- ν^+-invariant The concordance invariant $\nu^+ : \mathcal{C} \to \mathbb{Z}$ which refines τ was introduced by [Hom-Wu 2016]

Theorem (Hom-Wu 2016)

Let $K \subset S^3$ be an alternating knot, then

$$\nu^+(K) = \begin{cases} 0 & \sigma(K) \geq 0 \\ -\frac{1}{2}\sigma(K) & \sigma(K) < 0. \end{cases}$$

- ϵ-invariant The invariant $\epsilon(K) \in \{\pm 1, 0\}$ was introduced by [Hom 2014]. ϵ-invariant is a refinement of τ in the sense that $\epsilon(K) = 0 \Rightarrow \tau(K) = 0$.

Theorem (Hom 2014)

For alternating knot $K \subset S^3$,$\epsilon(K) = \mathrm{sgn}(\tau(K))(= -\mathrm{sgn}(\sigma(K))$.

- Υ-invariant [Ozsváth-Stipscitz-Szabó 2017] introduced a knot invariant $\Upsilon_K : [0, 2] \to \mathbb{R}$ using the filtered structure on *CFK*.

Theorem (Ozsváth-Stipscitz-Szabó 2017)

For alternating knot $K \subset S^3$, $\Upsilon_K(t) = (1 - |t - 1|)\frac{\sigma(K)}{2}$.

- s-invariant The concordance invariant $s : \mathcal{C} \to \mathbb{Z}$ was introduced by [Rasummusen 2004] using Khovanov homology.

Theorem (Rasummusen 2004)

For alternating knot $K \subset S^3$, $s(K) = \sigma(K)$.

- δ-invariant [Manolescu-Owens 2007] defined a knot concordance invariant $\delta(K) \in \mathbb{Z}$.
  Its refinement $\{\delta_j(K)\}_{j\in\mathbb{Z}}$ is introduced by [Baraglia-Hekmati 2021] using equivariant Seiberg-Witten-Floer cohomology.

Theorem (Manolescu-Owens 2007, Baraglia-Hekmati 2021)

For alternating knot $K \subset S^3$, $\delta_j(K) = \delta(K) = -\frac{1}{2}\sigma(K)$.

- κ-invariant A concordance invariant $\kappa(K) \in \mathbb{Q}$ is introduced by [Konno-Miyazawa-Taniguchi 2021] using Seiberg-Witten-Floer K-theory with involution.

Theorem (Konno-Miyazawa-Taniguchi 2021)

For a two-bridge knot $K \subset S^3$, $\kappa(K) = -\frac{1}{16}\sigma(K)$.

Question

Can we detect a linear independent family of alternating knots which are algebraically slice by Floer theoritic techniques?

Theorem (Daemi-Imori-Sato-Scaduto-Taniguchi 2021)

For each $m \geq 7$ the following sequence of two bridge knots $\{K_{m,n}\}_n$ are torsion in the algebraic concordance group $\mathcal{C}_{\mathrm{alg}}$ but linearly independent in the concordance group $\mathcal{C}$.

$$\{K_{m,n} := K(212mn - 68n + 53, -106m + 34)\}_{n \in \mathbb{Z}_{>0}}$$

for each $m \in \mathbb{Z}_{\geq 7}$.

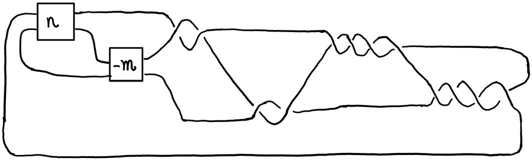


Figure: two-bridge knot $K_{m,n}$

Remark

The family $\{K_{m,n}\}_{n\in\mathbb{Z}>0}$ of two-bridge knots has the trivial Tristram-Levine signature if $m \geq 7$.

Corollary

For each $m \in \mathbb{Z}_{\geq 7}$ there is a sequence of integers l_n such that $\{l_n K_{m,n}\}_{n\in\mathbb{Z}_{\geq 7}}$ is algebraically slice but linear independent in $\mathcal{C}$.

Linear independent family of algebraically slice alternating knots can be detected by the Casson-Gordon invariant.

Question

Is there a linear independent family in

$$\{K(212mn - 68n + 53, -106m + 34)\}_{n\in\mathbb{Z}_{>0}}$$

with the trivial Casson-Gordon invariant?

- Satelite Operations
  Let $P \subset S^1 \times D^2$ be a knot in a solid torus. For any knot $K \subset S^3$, $P(K)$ denotes a knot obtained by the satellite operation of pattern P.

Remark

- *Satellite operation descends to the concordance group $P : \mathcal{C} \to \mathcal{C}$.*
- *In general, $P : \mathcal{C} \to \mathcal{C}$ is not homomorphism.*

- Satellite operations have been studied by in terms of structure of the concordance group.

Conjecture (Hedden-Caicedo)

The image of every non-constant satellite operation has infinite rank.

- "has infinite rank" means the image of satellite operation generates an infinite rank subgroup in $\mathcal{C}$.
- The problem can be separated whether the pattern P has winding number 0 or not.

Proposition (Hedden-Caicedo)

Let P be a pattern for satellite operation with non-zero winding number. Then $P : \mathcal{C} \to \mathcal{C}$ has infinite rank.

- The subtle case is when P has winding number 0. Hedden-Caicedo gave a partial answer to their conjecture.

Theorem (Hedden-Caicedo)

Let $P \subset S^1 \times D^2$ be a pattern with winding number 0 and $\Sigma_2(P(U))$ be the double branched cover along $P(U)$. If ∂D^2 has framed lifts to $\Sigma_2(P(U))$ with non-zero $\mathbb{Q}$-linking number, then $P : \mathcal{C} \to \mathcal{C}$ has infinite rank.

Question

Can we find other partial answer which does not covered by Hedden-Caicedo's theorem?

Let $P_{p,q} : \mathcal{C} \to \mathcal{C}$ be a map obtained by (p, q)-cable knot. $Wh_m : \mathcal{C} \to \mathcal{C}$ denotes the map obtained by m-twisted Whithead double.

Theorem (Daemi-Imori-Sato-Scaduto-Taniguchi 2021)

Let K_n be a sequence of twisted knots given in the figure. For any integer p, $\{P_{p,1} \circ Wh_m(K_n))\}_{n \in \mathbb{Z}>0}$ are linearly independent in the knot concordance group.

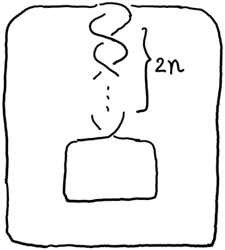


Figure: n-twisted knot K_n

Corollary

The satellite operator $P_{p,1} \circ Wh_m$ has infinite rank.

Remark

- *$P_{p,1} \circ Wh_m$ is a satellite operator with winding number 0.*
- *Since $\Sigma_2(P_{(p,1)} \circ Wh_m(U)) = \Sigma_2(U) = S^3$, $\mathbb{Q}$-linking number is trivial.*

In particular, our theorem gives another partial solution to the Hedden-Caiced's conjecture.

Remark

- *Images of $P_{p,1} \circ Wh_m$ are topologically slice.*
- *$g(P_{p,1} \circ Wh_m(K))$ are bounded by p.*