Successively almost positive links

Tetsuya Ito (Kyoto University)

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Diagrammatic class of links

For a class of diagram $\mathcal{C}$ (alternating, positive...), a link diagram D is k-almost $\mathcal{C}$ if the diagram is in the class $\mathcal{C}$ except k crossings;

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'Meta-Theorem/Motto'

Almost $\mathcal{C}$-links and $\mathcal{C}$-links often have the same/similar properties.

However, when k is large, k-almost $\mathcal{C}$ tells us nothing (every link diagram D is k-almost $\mathcal{C}$ when $k \geq c(D)$)

$\Rightarrow$ 'k-almost $\mathcal{C}$' is not a good generalization/notion.

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- To give a concise review of properties of (almost) positive links.
- To explain how naturally/easily these properties can be easily extend to (good) k-successively almost positive case.
- To show several improvement/new results of positive/almost positive links

$\Rightarrow$ Results on positive links usually do not use all crossings are positive !

Definition

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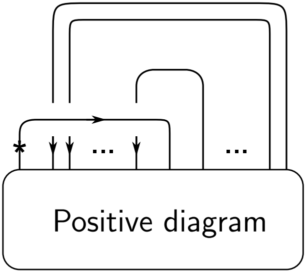
For $k \geq 0$, a link diagram D is *successively k-almost positive* if all but k crossings of D are positive, and the k negative crossings appear successively along a single overarc.



- succesively 0-almost positive = positive
  succesively 1-almost positive = almost positive
- succesively 2-almost positive $\neq$ 2-almost positive

Origin

This type of diagram appeared in previous joint work with K. Motegi, M. Teragaito;

Theorem (I.-Motegi-Teragaito '21)

If a knot K admits a positive diagram D with p crossings, $\pi_1(K(r))$ (the fundamental group fo the r-surgery on K) has a generalized torsion element whenever $r \geq p$.

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We noticed this theorem can be extended to;

Theorem (I.-Motegi-Teragaito '21)

If a knot K admits a successively k-almost positive diagram D with p crossings, $\pi_1(K(r))$ (the fundamental group fo the r-surgery on K) has a generalized torsion whenever $r \geq p - k$.

Non-negativity property

σ_ω: Levine-Tristram signature at $\omega \in \{z \in \mathbb{C} \mid |z| = 1\}$
$\nabla_K(z) = \sum_i a_i(K) z^i$: Conway polynomial

Theorem

If K is almost positive,

1. $\sigma_\omega(K) \leq 0$ (Przytycki-Taniyama '10).
2. $a_i(K) \geq 0$. (Cromwell '89)

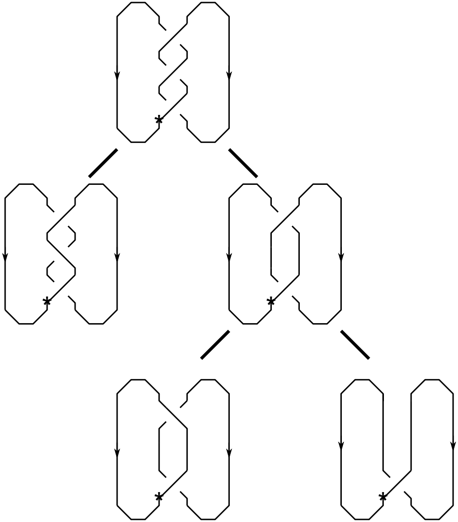
Theorem (I.)

If K is successively almost positive,

1. $\sigma_\omega(K) \leq 0$.
2. $a_i(K) \geq 0$.

Proof

Recall that one can construct *skein resolution tree* by trying to make D descending:

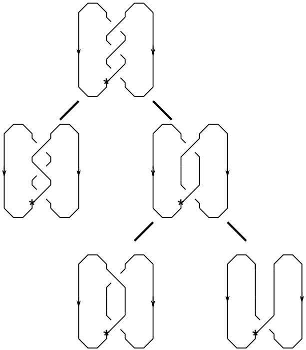


Proof

Since at the first k negative crossings are already descending, every skein resolution in the descending skein resolution tree is $+ \to -, 0$; By the property

$$\sigma_\omega(K_+) \leq \sigma_\omega(K_-), \nabla_{K_+}(z) = \nabla_{K_-}(z) + z\nabla_{K_0}(z)$$

we get desired results.



Good diagrammatic property

A positive diagram has the following nice properties:

- S_D: Canonical Seifert surface of D; Seifert surface obtained by Seifert's algorithm.

Theorem

If D is positive diagram of a link K,

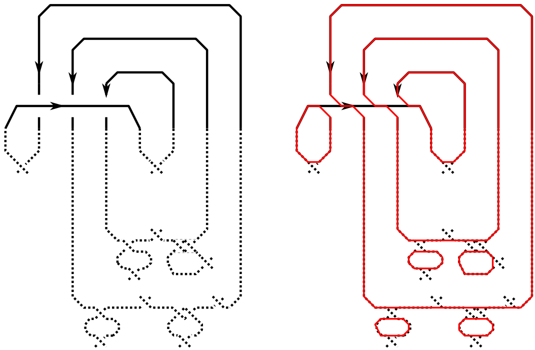
1. K is split link if and only if D is split (Easy exercise).
2. $\chi(S_D) = \chi(K)$. (Murasugi)

An almost positive diagram does not have the same property !

Good successively almost positive links

Definition

For $k \geq 0$, a link diagram D is good successively k-almost positive if it is successively k-almost positive and if two distinct Seifert circle s, s' of D are connected by a negative crossing then there are no other crossings connecting s and s'.

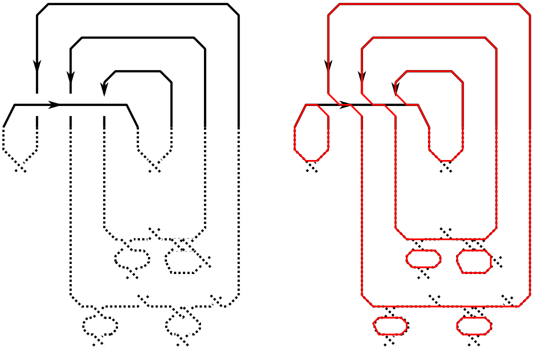


Good successively almost positive links

Theorem (I.)

If D is good successively almost positive diagram of a link K,

1. K is split link if and only if D is split.
2. $\chi(S_D) = \chi(K)$.



Properties of good successively almost positive links

It turns out good successively almost positive links share more properties with (almost) positive links;

Theorem (Cromwell '89)

If K is a non-split positive link, then

$$1-\chi(K)=\max\deg_z \nabla_K(z).$$

Theorem (I.)

If K is a non-split, good successively almost positive link, then

$$1-\chi(K)=\max\deg_z \nabla_K(z).$$

Properties of good successively almost positive links

Theorem (Stoimenow '05)

If K is an almost positive link. Then $\max\deg_z P_K(v,z) = 1-\chi(K)$.

Theorem (I.)

If K is a good successively almost positive link,
$\max\deg_z P_K(v,z) = \min\deg_v P_K(v,z) = 1-\chi(K)$.

Corollary

If K is an almost positive link,
$\max\deg_z P_K(v,z) = \min\deg_v P_K(v,z) = 1-\chi(K)$.

Fiberedness

Recall that when K is fibered,

- The Alexander polynomial is monic.
- The degree of the Alexander polynomial is equal to $1 - \chi(K)$.

Theorem (Cromwell '89, Stoimenow '05)

A (good almost) positive link K is fibered if and only if the Alexander polynomial is monic.

Theorem (I.)

A good successively almost positive link K is fibered if and only if the Alexander polynomial is monic.

Relation to other positivity

Theorem (Feller-Lewark-Lobb '18)

An almost positive link is strongly quasipositive.

Theorem (I.)

A good successively almost positive link is strongly quasipositive

Corollary

For a good successively almost positive link,

$$s(K) = 2\tau(K) = 2g_4(K) = 2g_3(K)$$

where $s(K)$ and $\tau(K)$ are the Rasumussen invariant and the Heegaard Floer τ-invariant.

Proofs

Proofs are done by the following line of arguments; assume that D is a good successively almost positive diagram.

1. Show $\chi(D) = \chi(K)$ and K is strongly quasipositive (manipulation of diagrams).
2. Using skein relations, prove $\max\deg_z \nabla_K(z) = 1 - \chi(D) = 1 - \chi(K)$.
3. Using skein relation and the fact $P_K(v,1) = \nabla_K(z)$, prove $\max\deg_z P_K(v,z) = 1 - \chi(K)$.
4. Using
   - Morton-Franks-Williams inequality: $SL(K) + 1 \leq \min\deg_v P_K(v,z)$
   - Inequality $\min\deg_v P_K(v,z) \leq \max\deg_z P_K(v,z)$
   - Property of strongly quasipositive links; $SL(K) = -\chi(K)$

   Prove $\min\deg_v P_K(v,z) = 1 - \chi(K)$

Signature estimate

Theorem (Baader-Dehornoy-Liechti '18)

If D is a positive diagram, then

$$\sigma(K) \leq \frac{1}{24}(\chi(S_D) - 1)$$

Corollary

For a general link diagram D

$$\sigma(K) \leq \frac{1}{24}(\chi(S_D) - 1) + 2c_{-}(D)$$

Is this best-possible ? As for positive braid link,

Theorem (Feller '18)

If K is a non-trivial positive braid link,

$$\sigma(K) \leq \frac{1}{8}(\chi(K) - 1)$$

Signature estimate

Theorem (I.)

For a reduced link diagram D of a non-trivial link

$$\sigma(K) \leq \frac{1}{12}(\chi(S_D) - 1) + \frac{4}{3}c_{-}(D) - \frac{1}{2}$$

Corollary (I.)

If K is a non-trivial positive link

$$\sigma(K) \leq \frac{1}{12}(\chi(K) - 1) - \frac{1}{2}$$

Conjecture (Feller '15)

If K is a non-trivial positive braid link,

$$\sigma(K) \leq \frac{1}{2}(\chi(K) - 1)$$

How about positive case ? (seemingly $\frac{1}{12}$ is not best-possible)

Sketch of proof

A proof is a direct adaptation of Baader-Dehornoy-Liechti's argument for general link diagrams (with some minor improvements):

- Consider the checkerboard surfaces B and W of D.
- Gordon-Litherland theorem asserts $\sigma(K)$ is determined by
  - the Gordon-Litherland form on $H_1(B)$ ($H_1(W)$); a generalization of Seifert form for non-orientable settings.
  - the boundary slope information of B, W (easily computed from the diagram)
- By an analysis of diagram find a large subspace $H_1(B)$, $H_1(W)$ having definite Gordon-Litherland form
  (caution; here we use four-color theorem to utilize planar feature of diagrams)
- $\Rightarrow$ estimate of $\sigma(K)$

Concordance finiteness

As an application of signature estimates,

Theorem (Baader-Dehornoy-Liechti '18)

Every topological concordance class $\mathcal{K}$ contains finitely many positive knots.

For a topological concordance class $\mathcal{K}$, let

$$g_c^{top}(\mathcal{K}) = \min\{g(K) \mid K \in \mathcal{K}\}$$

be the topological concordance genus.

Corollary (I.)

For $0 < d < \frac{1}{9}$, every topological concordance class $\mathcal{K}$ contains finitely many successively k-almost positive knots such that $k \leq d g_c^{top}(\mathcal{K})$.

Interesting identity for positive links

$D_K(a,z)$: Dubrovnik version of the Kauffman polynomial
$P_K(v,z)$: HOMFLY polynomial
Let us put

$$D_K(v,z) = \sum_i D_K(z;i)a^i, P_K(v,z) = \sum_i P_K(z;i)v^i$$

Theorem (Yokota '12)

When K is positive,

$$D_K(z;\chi(K)-1) = P_K(z;\chi(K)-1)$$

is a non-trivial, non-negative (i.e. coefficients are non-negative) polynomial.

Not-good case

An almost positive diagram D is

- of type I if D is good successively 1-almost positive.
- of type II otherwise.

This dichotomy plays a fundamental role in a study of almost positive links.

Theorem (I.)

If K is almost positive link of type II, then

$$D_K(z;\chi(K)-1) = P_K(z;\chi(K)-1)$$

is a non-trivial, non-negative (i.e. coefficients are non-negative) polynomial.

(c.f.) Theorem (Tagami '19)

If K is almost positive link of type II, then K is Lagrangian fillable.

Proof

The proof is immediate once we understand the reason why $D_K(z;\chi(K)-1) = P_K(z;\chi(K)-1)$ is true;

- Positive diagram can be regarded as a *front diagram* $\mathcal{D}$, a diagram representing Legendrian link (Tanaka '99).
- $D_K(z;\chi(K)-1)$ is regarded as an unoriented ruling polynomials, certain (graded) counts of 'ruling' of $\mathcal{D}$.
- Similarly, $P_K(z;\chi(K)-1)$ is regarded as an oriented ruling polynomials, certain (graded) counts of 'oriented ruling' of $\mathcal{D}$ (Rutherford '06).
- For a front diagram $\mathcal{D}$ from positive diagram,
  - every ruling is an oriented ruling.
  - (oriented) ruling exists.

We can apply a similar argument when D is almost positive of type II.

Confession

You may feel that we get a new interesting class of links. However...

Confession of sin

We do not give any examples of knots which are (good) successively positive but not almost positive.

- It is easy to find/construct candidates.
- Since (good) successively almost positive links shares most properties of almost positive links), there is no easy way to check a candidate is indeed not almost positive...

Remark

It is not so easy and obvious to distinguish positive links with almost positive links.

Good news

After putting the paper on arXiv, A. Stoimenow informed me;

Theorem (Stoimenow)

There exists a knot (17 crossings) which is good successively 2-almost positive but is not almost positive.

$\Rightarrow$ 'Good successively almost positive' is a genuine generalization of (almost) positive links !
$\Rightarrow$ Several interesting problems...

- To what extent one can extends properties of good successively almost positive to successively almost positive ?
- Try to extend/find properties of (good) successively almost positive links.

References

This talk is based on my preprint

Tetsuya Ito,
Successively almost positive links,
arXiv:2111.14361 (to be updated soon)