On keen bridge splitting of links

(joint work with Ayako Ido and Yeonhee Jang)

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結び目の数理 IV

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Part 1

(Strongly) keen Heegaard splitting

(Hempel) distance of Heegaard splittings

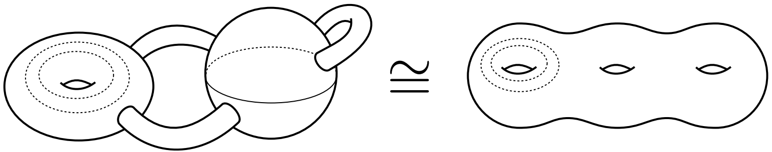
C : *compression-body* if
it is a connected 3-mfd. obtained from $S \times [0,1] \cup$ (3-ball)
(S : closed orientable surface) by attaching " 1-handles " to
$S \times \{1\} \cup \partial$(3-ball).

Notation:

$\partial_- C := S \times \{0\}$,
$\partial_+ C := \partial C \setminus \partial_- C$,

The genus of $\partial_+ C$ is called the *genus* of the compression-body C.

A compression body C is called a *handlebody* if $\partial_- C = \emptyset$.



Heegaard splittings

M : compact orientable 3-manifold

Heegaard splittings

$C_1 \cup_\Sigma C_2$: (genus-g) *Heegaard splitting* of M (Σ: *Heegaard surface*) if
C_i : *genus-g compression-body* $(\subset M)$
s.t. $C_1 \cup C_2 = M$, *and* $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = \Sigma$.

It is known that any M admits a Heegaard splitting.

An alternate expression of H.S.

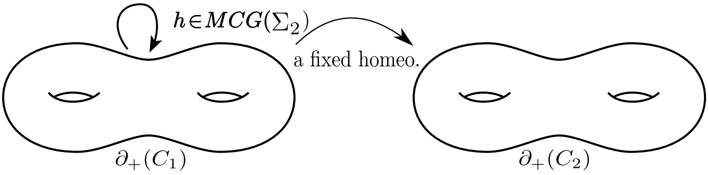
$\Sigma = \Sigma_g$: orientable surface of genus $g \geq 2$.

MCG(Σ): *Mapping class group* of Σ: the group of automorphisms of Σ.

An alternate expression of the H.S. $C_1 \cup_\Sigma C_2$ is:

$$C_1 \cup_\Sigma C_2 = C_1 \cup_h C_2,$$

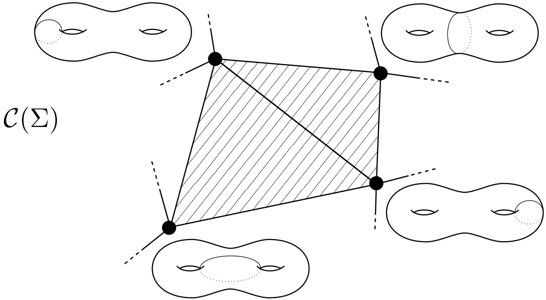
where $h : \Sigma(= \partial_+ C_1) \to \Sigma(= \partial_+ C_1) \in \mathrm{MCG}(\Sigma)$



Curve complex: W.Harvey

The *curve complex* $C(\Sigma)$ is a simplicial complex defined as follows:

0-simplex $\leftrightarrow$ (isotopy class of) an essential simple closed curve on Σ,
n-simplex $\leftrightarrow$ disjoint non-parallel $n+1$ simple closed curves on Σ.



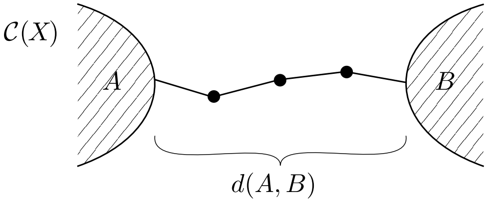
W. J. Harvey. Boundary structure of the modular group. In I. Kra and B. Maskit, ed., Riemann Surfaces and Related Topics: Proc. of the 1978 Stony Brook Conf., vol. 97, Ann. of Math. Stud. Princeton,

Distance in the curve complex

For vertices a and b ($\in \mathcal{C}(\Sigma)^{(0)}$),
$d(a,b) = d_{\mathcal{C}(\Sigma)}(a,b) :=$ (the smallest number of 1-simplexes in a path connecting a and b in $\mathcal{C}(\Sigma)$).

A shortest path $[a_0, a_1, a_2, \ldots, a_n]$ connecting two vertices a_0 and a_n is called a *geodesic*.

For sets A and $B (\subset \mathcal{C}(\Sigma)^{(0)})$,
$d(A,B) = d_{\mathcal{C}(\Sigma)}(A,B) := \min\{d(a,b) | a \in A, b \in B\}$.



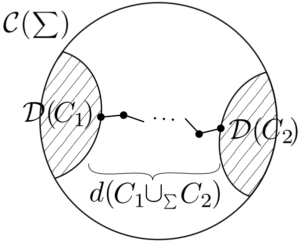
Distance of Heegaard splitting

Let:

M: compact orientable 3-manifold
$C_1 \cup_\Sigma C_2$: Heegaard splitting of M
$\mathcal{C}(\Sigma)$: curve complex of Σ

$\mathcal{D}(C_i)$ ($\subset \mathcal{C}(\Sigma)^{(0)}$): disk complex of C_i is defined as:

$$a \in \mathcal{D}(C_i) \Leftrightarrow a \text{ bounds a disk in } C_i.$$



Definition (Distance of Heegaard splitting)

$$d(C_1 \cup_\Sigma C_2) := d_{\mathcal{C}(\Sigma)}(\mathcal{D}(C_1), \mathcal{D}(C_2))$$
$$\left(= \min\{d(a,b) \mid a \in \mathcal{D}(C_1), b \in \mathcal{D}(C_2)\} \right)$$

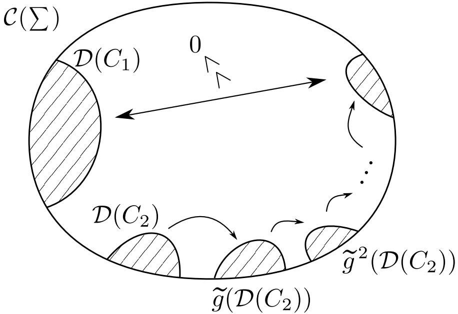
J.Hempel, 3-manifolds as viewed from the curve complex, Topology 40 (2001) 631-657

High distance Heegaard splitting

Theorem(Hempel)

$\exists g \colon \Sigma \to \Sigma$ s.t. $d(C_1 \cup_{g^n} C_2) \to \infty$ $(n \to \infty)$

Idea of the proof: $h \in \mathrm{MCG}(\Sigma)$ induces a bijection on $\mathcal{C}(\Sigma)^{(0)}$, and it naturally extends to an isometry on $\mathcal{C}(\Sigma)$ denoted by $\tilde{h}$.
Then there is $g \in \mathrm{MCG}(\Sigma)$ which is a so-called pseud-Anosov satisfying:



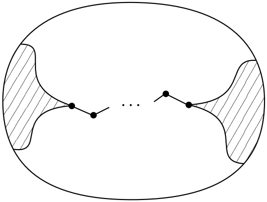
(Strongly) keen Heegaard splitting

Definition (keen Heegaard splitting)

$C_1 \cup_\Sigma C_2$: *keen*. $\overset{}{\underset{\text{def.}}{\Longleftrightarrow}}$ $d(C_1 \cup_\Sigma C_2)$ is realized by unique pair of elements of $\mathcal{D}(C_1)$ and $\mathcal{D}(C_2)$.

Definition (Strongly keen Heegaard splitting)

$C_1 \cup_\Sigma C_2$:*strongly keen* $\underset{\text{def.}}{\Longleftrightarrow}$ $C_1 \cup_\Sigma C_2$ is keen, and $d(C_1 \cup_\Sigma C_2)$ is realized by unique geodesic.



(Strongly) keen Heegaard splitting

Iguchi-Koda show that the Goeritz group of (strongly/weakly) keen Heegaard surface is a restricted from.

> *D. Iguchi and Y. Koda, Twisted book decompositions and the Goeritz groups, Topology Appl., 272 (2020), 107064.*

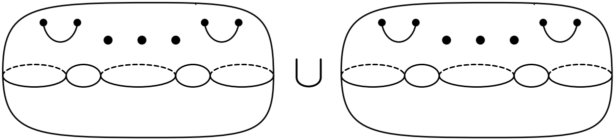
Part 2

(Strongly) keen bridge splitting

Bridge splitting (B.S.)

M: closed 3-mfd.
L: link in M
$(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$(or (L, F)): (g, b)-splitting of (M, L) (or, simply L) if:

- $V_1 \cup_F V_2$: genus g H.S. of M,
- $P = L \cap F$, and
- $t_i = L \cap V_i$: b arcs simultaneously parallel to ∂V_i $(i = 1, 2)$.



$(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$

(g, b)- splitting

Curve complex ($C(S \setminus \{\text{punctures}\})$)

Let

S: genus g, ori. surface with c boundary comp., and p punctures

α ($\subset S \setminus \{\text{punctures}\}$): a simple closed curve (s.c.c.)

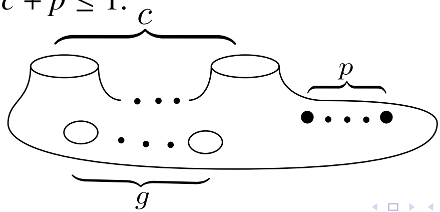
- α is *inessential* if

  α cuts off a disk with at most one punctures, or

  α is paralle to a component of ∂S.
- α is *essential* if it is not inessential.
- We say that S is *sporadic* if

  $g = 0$, $c + p \leq 4$, or

  $g = 1$, $c + p \leq 1$.



Curve complex ($\mathcal{C}(S \setminus \{\text{punctures}\})$)

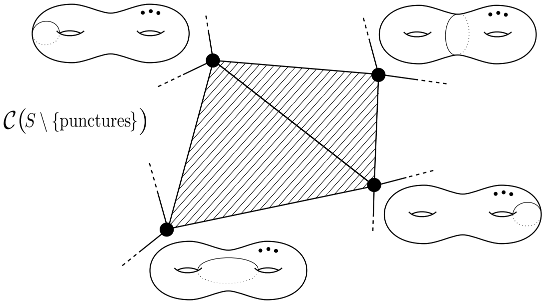
Suppose $S \setminus \{\text{punctures}\}$ is not sporadic.

Curve complex

The *curve complex* $\mathcal{C}(S \setminus \{\text{punctures}\})$ is a simplicial complex s.t.

0-simplex $\leftrightarrow$ (isotopy class of) an ess. s.c.c. on $S \setminus P$,

n-simplex $\leftrightarrow$ disjoint non-parallel $n+1$ ess. s.c.c. on $S \setminus P$.



Distance of bridge splitting

Let:

$(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$: (g, b)-splitting of L

- $C(F \setminus P)$: curve complex of $F \setminus P$
- $\mathcal{D}(V_i \setminus t_i)$ ($\subset C(F \setminus P)^{(0)}$): disk complex of $V_i \setminus t_i$ is defined as:

$$a \in \mathcal{D}(V_i \setminus t_i) \;\Leftrightarrow\; a \text{ bounds a disk in } V_i \setminus t_i.$$

Definition (Distance of bridge splitting)

$$\begin{aligned} d((V_1, t_1) \cup_{(F,P)} (V_2, t_2)) \quad &:= \quad d_{C(F\setminus P)}(\mathcal{D}(V_1 \setminus t_1), \mathcal{D}(V_2 \setminus t_2)) \\ &\left(= \min\{d(a,b) | a \in \mathcal{D}(V_1 \setminus t_1), b \in \mathcal{D}(V_2 \setminus t_2)\} \right) \end{aligned}$$

Some researches on distance of B.S.

Existence of high distance B.S.

- T.Saito, Genus one 1-bridge knots as viewed from the curve complex, Osaka J. Math. 41(2004), 427-454
- Campsi-Rathbun, High distance knots in closed 3-manifolds, JKTR, 21(2002)
- Blair-Tomova-Yoshizawa, High distance bridge surfaces, AGT 13(2013), 2925-2946.
- Ichihara-Saito, Knots with arbitrarily high distance bridge splittings, Bull.Korean Math Soc., 50(2013), 1989-2000

Some researches on distance of B.S.

Upper bound of distance of B.S.

- Bachman-Schleimer, Distance and bridge position, Pacific J.Math. 219(2005), 221-235
  ($^\exists S$: ess. surface in the exterior of $L \Rightarrow d(L, F) \leq -\chi(S) + 2$)
- M.Tomova, Multiple bridge surfaces restrict knot distance, AGT 7(2007), 957-1006
  (F, F': distinct bridge surfaces of $L \Rightarrow d(L, F) \leq -\chi(F' \setminus L) + 2$)
- Y.Jang, Distance of bridge surfaces for links with essential meridional spheres, Pacific J.Math. 267(2014), 121-130
  ($^\exists S$: ess. $n(\geq 4)$ punctured sphere in the exterior of $L \Rightarrow$ $d(L, F) \leq n - 2(= -\chi(S))$)
- A.Ido, An estimate of Hempel distance for bridge spheres, Bull.Korean Math Soc., 52(2015), 735-740
  (S, S': distinct bridge spheres of $L \Rightarrow d(L, F) \leq -\chi(S' \setminus L)$)

(Strongly) keen bridge splitting

Let

$(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$: (g, b)-splitting of L

Definition (keen bridge splitting)

$(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$ is *keen* $\underset{\text{def.}}{\Longleftrightarrow}$ $d(L, F)$ is realized by unique pair of elements of $\mathcal{D}(V_1 \setminus t_1)$ and $\mathcal{D}(V_2 \setminus t_2)$

Definition (Strongly keen bridge splitting)

$(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$:*strongly keen* $\underset{\text{def.}}{\Longleftrightarrow}$ $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$ is keen, and $d(L, F)$ is realized by unique geodesic.

Main Results

Theorem 1(Ido-Jang-K.)

$\forall g \geq 0$, $\forall b \geq 1$, and $\forall n \geq 1$
except for $(g, b) = (0, 1)$, and $(g, b, n) = (0, 1, 3)$
$^{\exists}$ strongly keen (g, b)-splitting with distance n.

Remark(Saito)

If $(g, b) = (0, 1)$, then $C(F \setminus P) = \emptyset$.
If $(g, b, n) = (1, 1, 1)$, then the ambient mfd. is $S^2 \times S^1$, and the link is a core knot.

T.Saito, Genus one 1-bridge knots as viewed from the curve complex, Osaka J.Math. 41(2004), 427-454.

Main Results

For the case when $(g, b, n) = (0, 3, 1)$, we have:

Theorem 2(Ido-Jang-K)

"Any" $(0, 3)$-splitting with distance 1 cannot be keen.

Remark

$\exists_{\infty}(0, 3)$-splitting with distance 1.

In fact,

- L admits $(0, 3)$-splitting with distance 1

  $\Leftrightarrow$

- L: connected sum of two 2-bridge links, which is non-separable

Important remark

Remark

In

> Ido-Jang-K.: Bridge splittings of distance exactly n, Top. and App. 196(2015), 1395-1411

we "showed" that

> for each $n \geq 2$, $g \geq 0$, and $n \geq 2$ (except $(g, b) = (0, 1), (0, 2)$), ${}^{\exists}(g, b)$-splitting with distance n.

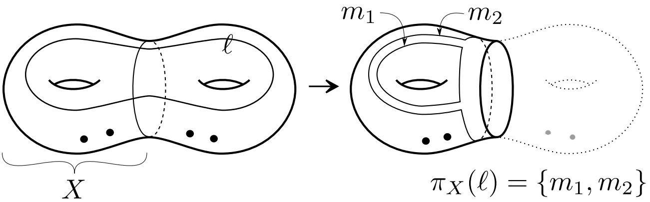
However we found that the proof contains an error. We note that Theorem 1 recovers the above mentioned result.

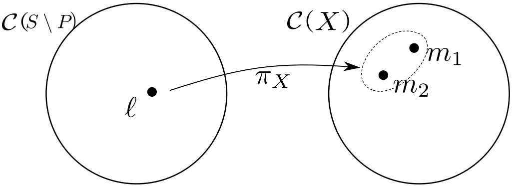
Part 3

Ingredient and recipe of the proof

Ingredient: Subsurface projection

X : essential non-simple subsurface of Σ,
$2^{C(X)^{(0)}}$: the power set of $C(X)^{(0)}$.
Subsurface projection $\pi_X : C(\Sigma)^{(0)} \to 2^{C(X)^{(0)}}$ is defined by





Ingredient: Subsurface projection

Lemma 1 (π_X is 2-Lipschitz:Masur-Minsky'00)

π_X is 2-Lipschitz, hence,
$\forall [a_0, a_1, \ldots, a_m]$ a geodesic in $C(S \setminus P)$ such that

$$a_i \cap X \neq \emptyset \ (i = 0, 1, \ldots, m),$$

We have:

$$\mathrm{diam}_{C(X)}(\pi_X(a_0) \cup \cdots \cup \pi_X(a_m)) \leq 2m.$$

Lemma 2 (Masur-Minsky: Prop. 4.6, 1999)

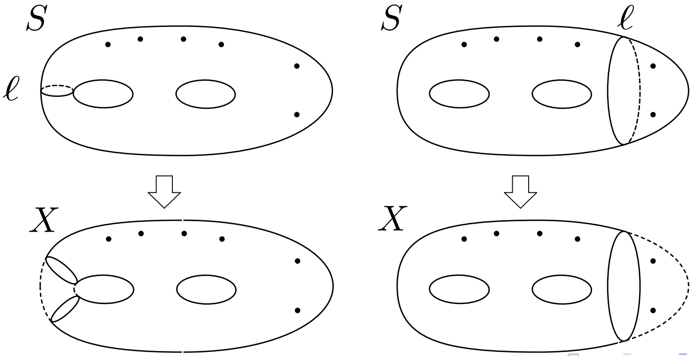
Suppose $S \setminus P$ is non-sporadic. Then ${}^{\exists}C > 0$, s.t.

$$\forall \text{ p.A. } h \in \mathrm{MCG}(C(S \setminus P)), \forall \gamma \in C(S \setminus P), \forall n \in \mathbf{Z},$$

$$d_{C(S \setminus P)}(h^n(\gamma), \gamma) \geq c|n|.$$

Recipe: Unique geodesic

Let
ℓ: ess. s.c.c. in $S \setminus P$, where $P = \{\text{punctures}\}$
Suppose: ℓ is non-sep. in S or ℓ cuts off a disk with 2 punctures.
Then the subsurface associated with ℓ (denoted X) is defined as follows.



Recipe: Unique geodesic

Remark (Forcing s.c.c.)

In the setting of the last slide, we note:

If m ($\in C^{(0)}(S \setminus P)$) is disjoint from X, then $m = \ell$.

Recipe: Unique geodesic

Proposition (Extending geodesic with uniqueness: Ido-Jang-K.)

Let $[\ell_0, \ell_1, \dots, \ell_{n-1}, \ell_n]$ be a path in $C(S \setminus P)$
Assume that the following conditions are satisfied
(1) $[\ell_0, \ell_1, \dots, \ell_{n-1}]$ is the unique geodesic connecting ℓ_0 and ℓ_{n-1},
(2) ℓ_{n-1} is non-sep. in S, or cuts off a twice punctured disk from $S \setminus \{\text{punctures}\}$, and
(3)

$$\mathrm{diam}_{C(S\setminus P)}(\pi_X(\ell_0), \pi_X(\ell_n)) > 2n,$$

where X is the surface associated to ℓ_{n-1}.

Then $[\ell_0, \ell_1, \dots, \ell_{n-1}, \ell_n]$ is the unique geodesic connecting ℓ_0 and ℓ_n.

Remark

Suppose: we have already obtained $[\ell_0, \ell_1, \dots, \ell_{n-1}]$.
Then by using Lemma 2, we can obtain ℓ_n satisfying (2) and (3).

Outline of the proof of Prop.

Let $[m_0 = \ell_0, m_1, \ldots, m_{n-1}, m_n = \ell_n]$ be a path connecting ℓ_0 and ℓ_n.

Claim. $^{\exists}i$ s.t. $m_i = \ell_{n-1}$

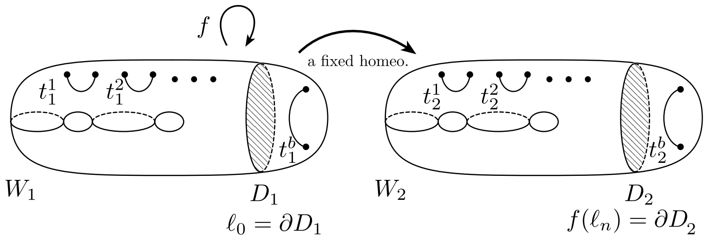
Proof. Assume no such i. Then by (2) each m_j intersects X.
This fact together with Lemma 1 implies

$$\mathrm{diam}_{C(S \setminus P)}(\pi_X(\ell_0), \pi_X(\ell_n)) \leq 2n, \text{contradicting (3)}$$

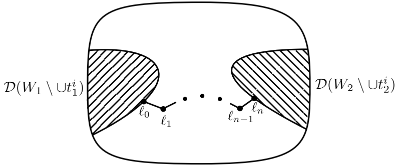
By (1) and Claim, we can show that Prop. holds.

Construction of strongly keen bridge splitting

Let $[\ell_0, \ell_1, \ldots, \ell_n]$ be a geod. as in Prop. Take a homeo.
$f : \partial W_1 \setminus (\cup t_1^j) \to \partial W_1 \setminus (\cup t_1^j)$ s.t.

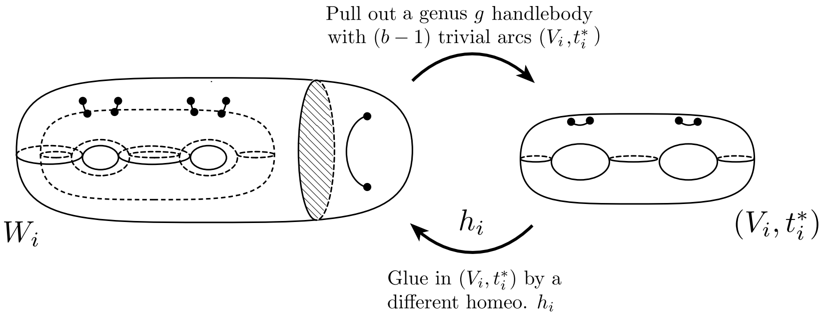


The disk complexes in the curve complex will look as :



Construction of strongly keen bridge splitting

Then we modify W_i as follows:



Construction of strongly keen bridge splitting

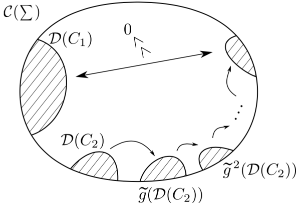
Ichihara-Saito gave a variation of Hempel's argument.

Ichihara-Saito, Knots with arbitrarily high distance bridge splittings, Bull.Korean Math Soc., 50(2013), 1989-2000

Theorem(Hempel)

$\exists g \colon \Sigma \to \Sigma$ s.t. $d(C_1 \cup_{g^n} C_2) \to \infty$ $(n \to \infty)$

Idea of the proof: $h \in \mathrm{MCG}(\Sigma)$ induces a bijection on $\mathcal{C}(\Sigma)^{(0)}$, and it naturally extends to an isometry on $\mathcal{C}(\Sigma)$ denoted by $\tilde{h}$.
Then there is $g \in \mathrm{MCG}(\Sigma)$ which is a so-called pseud-Anosov satisfying:



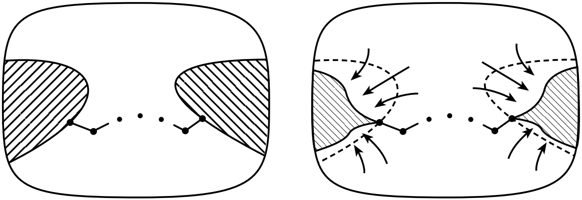
Construction of strongly keen bridge splitting

By using the argument , we can take h_i so that

$$d_{\partial V_1 \setminus t_1^*}(\ell_1, h_1(\mathcal{D}(V_1 \setminus t_1^*))) > 2$$

$$d_{\partial V_2 \setminus t_2^*}(\ell_{n-1}, h_2(\mathcal{D}(V_2 \setminus t_2^*))) > 2$$

These conditions change the disk complexes in the curve complex as follows.



This gives a strongly keen B.S.