シャドウ補空間の基本群の表示

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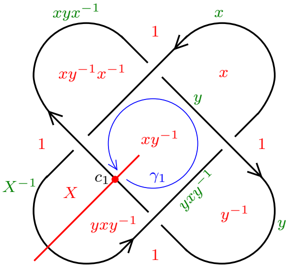
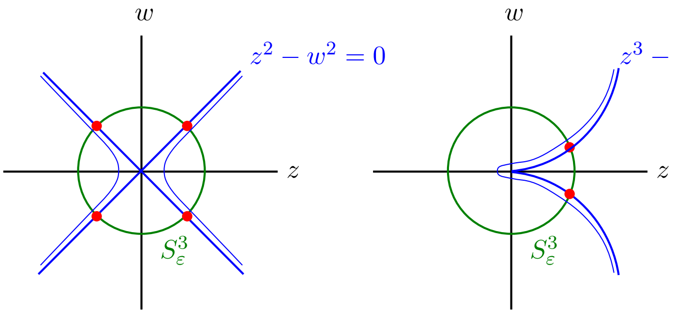


joint work with Yuya Koda and Hironobu Naoe

Contents

$f : \mathbb{C}^2 \to \mathbb{C}$ **a polynomial map s.t.** $f(0) = f_x(0) = f_y(0) = 0.$

Examples: $f(z,w) = z^2 - w^2$ **(left),** $f(z,w) = z^3 - w^2$ **(right)**

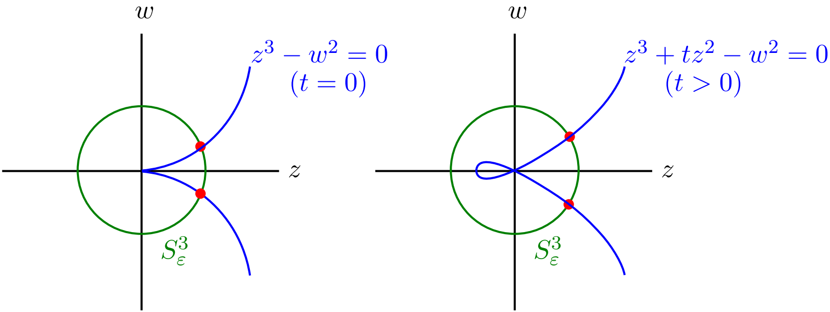


Cone Lemma (Milnor, '68)

$(B_\varepsilon, B_\varepsilon \cap f^{-1}(0))$ **is homeomorphic to Cone**$(S^3_\varepsilon, S^3_\varepsilon \cap f^{-1}(0))$.

Real deformation

Example: $f_t(z,w) = z^3 + tz^2 - w^2$

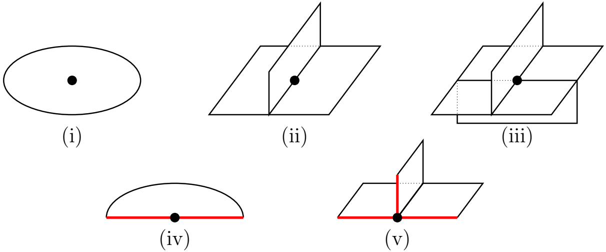


Theorem (A'Campo, Gusein-Zade, '74-75)

Any complex plane curve singularity has a real deformation.

Definition

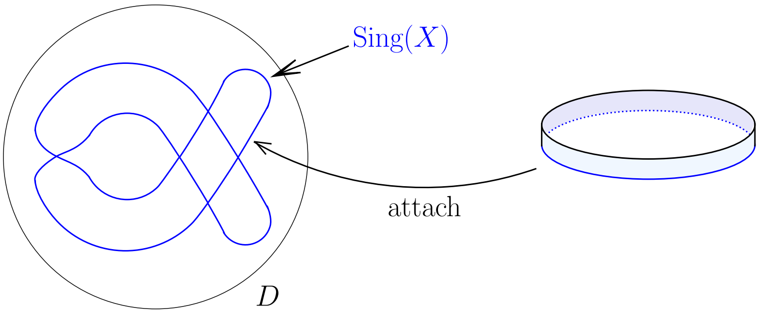
If each point of a compact space X has a neighborhood homeomorphic to one of (i)-(v) in the figure, then X is called a *simple polyhedron*.



§2. Shadows with immersed curve presentations

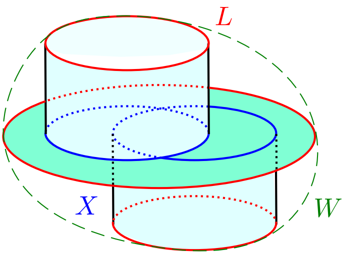
Definition

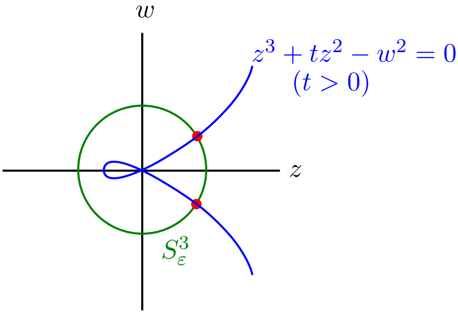
A simple polyhedron X is said to have an *immersed curve presentation* if there exists a disk D in X such that $X \setminus D$ is a disjoint union of copies of $S^1 \times (0, 1]$ whose closures do not intersect the boundary ∂D of D.



Definition

Let W be a compact, oriented, smooth 4-manifold with boundary ∂W and L be a link in ∂W. If a handle decomposition of W does not contain 3- or 4-handles, W collapses onto a locally-flat simple polyhedron X with $\partial X = L$. Such a polyhedron is called a *shadow* of (W, L).

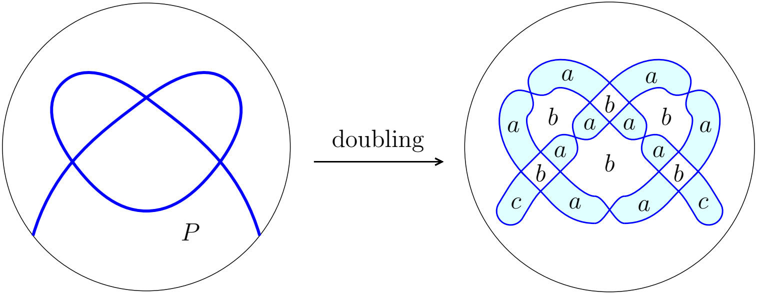




§2. Shadows with immersed curve presentations

Theorem (I.-Naoe, '20)

A fiber surface of a *divide* P is represented by the immersed curve presentation obtained from P by doubling it as in the figure and assigning *gleams* as $a = \frac{1}{2}, b = -1$ and $c = 0$.

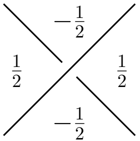


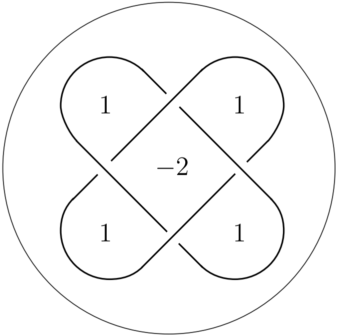
Note: A real deformed curve of a plane curve singularity is a divide.

Sum of local contributions

C_X: an immersed curve presentation

We add over/under information to each double point of C_X arbitrarily. For each internal region R of X, let $c(R)$ be the *sum of local contributions* at the vertices of X on the boundary of R given as in the figure.



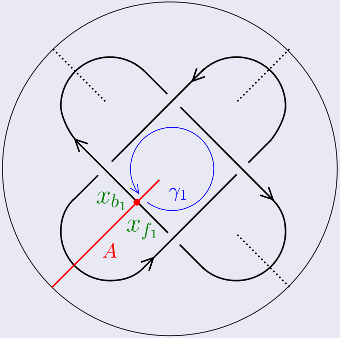


§3. Presentation of fundamental groups

Theorem (I.-Koda-Naoe, '21)

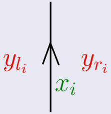
Let Y be a simple subpolyhedron of $(X, \mathfrak{gl})$ not intersecting ∂D and A be a *system of cutting trees* of C_X. Then

$$\pi_1(B^4 \setminus Y) \cong \langle x_1, \ldots, x_n, y_1, \ldots, y_m \mid s_1, \ldots, s_{n'}, t_1, \ldots, t_{n''} \rangle,$$



where x_k is the meridian of a strand of $C_X \setminus A$, y_k is the meridian of a region of $Y \setminus A$ on D, and

- $s_i = y_{r_i} x_i y_{l_i}^{-1}$,
- $t_j = x_{b_j}^{-1} \gamma_j x_{f_j} \gamma_j^{-1}$,
- $\gamma_j = y_{j_k}^{\mathfrak{gl}(R_{j_k}) - c(R_{j_k})} \cdots y_{j_2}^{\mathfrak{gl}(R_{j_2}) - c(R_{j_2})} y_{j_1}^{\mathfrak{gl}(R_{j_1}) - c(R_{j_1})}$.



- $s_i = y_{r_i} x_i y_{l_i}^{-1}$
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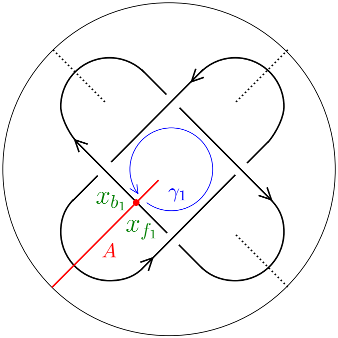
- $\gamma_j = y_{j_k}^{\mathfrak{gl}(R_{j_k})-c(R_{j_k})} \cdots y_{j_2}^{\mathfrak{gl}(R_{j_2})-c(R_{j_2})} y_{j_1}^{\mathfrak{gl}(R_{j_1})-c(R_{j_1})}$

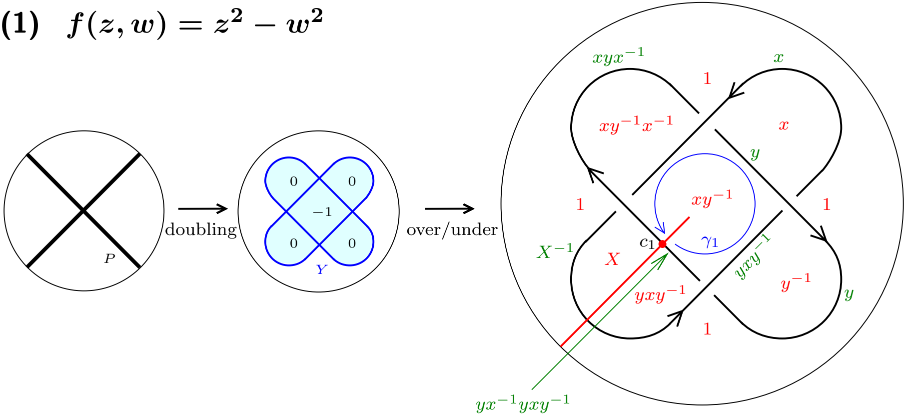
Theorem (I.-Koda-Naoe, '21)

If Y is the union of the regions of X on D not intersecting ∂D and $\mathfrak{gl}(R) - c(R) = 0$ for any region R of Y on D then the presentation in the previous theorem becomes the Wirtinger presentation of C_X with the over/under information.



§4. Examples

(1) $f(z,w) = z^2 - w^2$

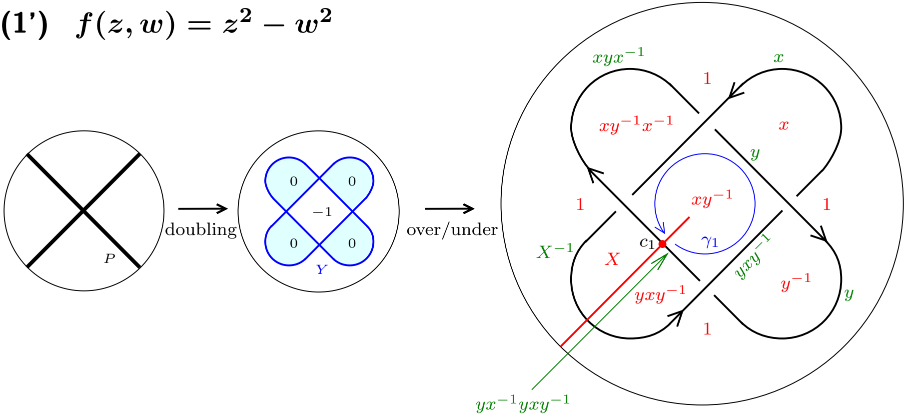


$$\gamma_1 = (xy^{-1})^{-1-(-2)} = xy^{-1}$$

$$xyx^{-1} = (xy^{-1})(yx^{-1}yxy^{-1})(xy^{-1})^{-1} = xy^{-1}yx^{-1}yxy^{-1}yx^{-1}$$

Hence, $\pi_1(B^4 \setminus Y) \cong \mathbb{Z}\langle x\rangle \oplus \mathbb{Z}\langle y\rangle$**.**

(1') $f(z,w) = z^2 - w^2$



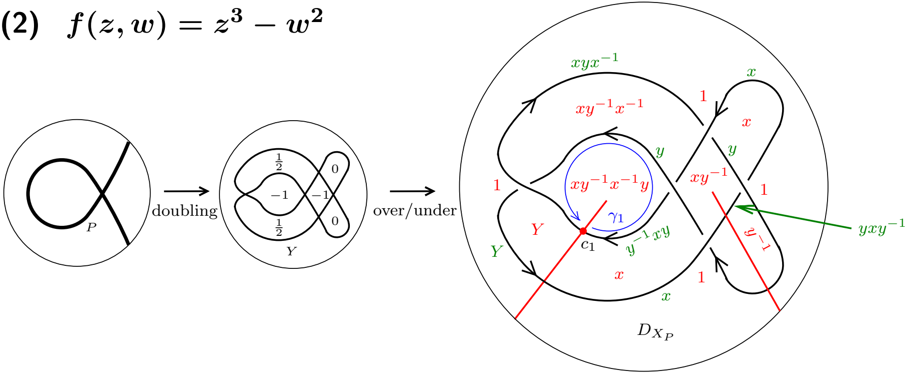
$\gamma_1 = (xy^{-1})^{-1-(-2)} = xy^{-1}$

$xyx^{-1} = (xy^{-1})(yx^{-1}yxy^{-1})(xy^{-1})^{-1} = xy^{-1}yx^{-1}yxy^{-1}yx^{-1}$

And $xy^{-1} = 1$. Hence, $\pi_1(B^4 \setminus Y) \cong \mathbb{Z}$. ($= \pi_1(B^4 \setminus$ Milnor fiber$)$)

§4. Examples

(2) $f(z,w) = z^3 - w^2$

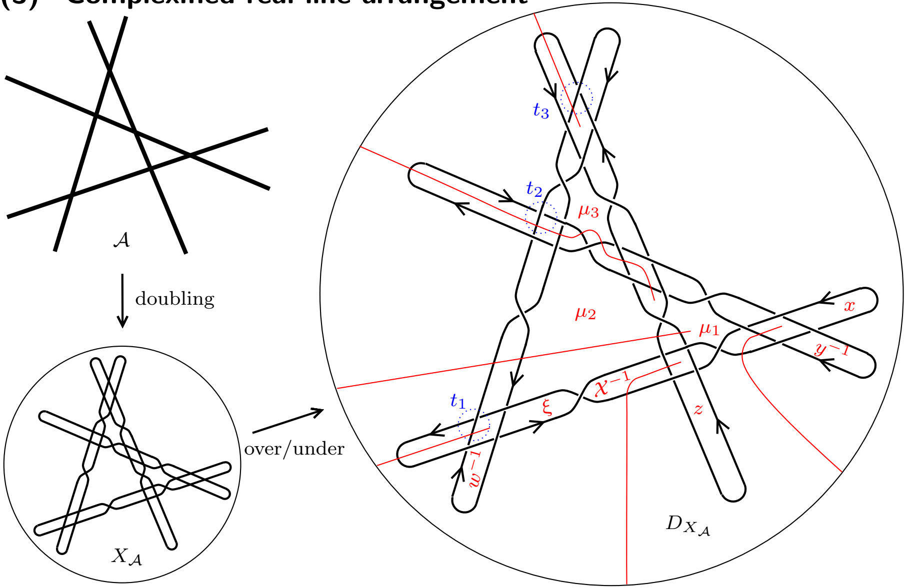


$$\gamma_1 = (xy^{-1}x^{-1}y)^{-1-(-1)} = 1$$

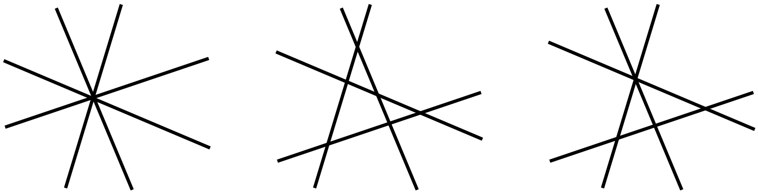
$$xyx^{-1} = y^{-1}xy$$

Hence, $\pi_1(B^4 \setminus Y) \cong \langle x, y \mid xyx = yxy \rangle$.

(3) Complexified real line arrangement



(3) Complexified real line arrangement



$$\pi_1(\mathbb{C}^2 \setminus \mathcal{A}'_{\mathbb{C}}) \cong \langle x, y, \mathcal{Z}, \mathcal{W} \mid x\mathcal{Z}y\mathcal{W} = \mathcal{Z}y\mathcal{W}x = y\mathcal{W}x\mathcal{Z} = \mathcal{W}x\mathcal{Z}y \rangle$$

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{A}_{\mathbb{C}}) \cong \mathbb{Z}\langle x\rangle \oplus \mathbb{Z}\langle y\rangle \oplus \mathbb{Z}\langle z\rangle \oplus \mathbb{Z}\langle w\rangle$$

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{A}''_{\mathbb{C}}) \cong \mathbb{Z}\langle x\rangle \oplus \langle y, z, w \mid wzy = ywz = zyw \rangle$$

A presentation of the fundamental group of the complement of a complexified real line arrangement had been given by Randell ('85).

The details can be found in the preprint

M. Ishikawa, Y. Koda, H. Naoe,
Presentation of the fundamental groups of complements of shadows,
arXiv:2110.02431 [math.GT]

Thank you for your attention !