An irreducible rectangle tiling contains a spiral

Tomoe Motohashi
Department of Network and Multi-Media Engineering
College of Engineering
Kanto Gakuin University
1-50-1 Mutuura Higashi, Kanazawa-ku
Yokohama, 236-8501, Japan
tomoe@kanto-gakuin.ac.jp

Kouki Taniyama
Department of Mathematics
School of Education
Waseda University
1-6-1 Nishi-Waseda, Shinjuku-ku
Tokyo, 169-8050, Japan
taniyama@waseda.jp

Abstract

We consider a tiling of a square by finitely many tiles each of which is a rectangle. We do not assume that the tiles are mutually congruent. Such a tiling is called *irreducible* if for any two tiles the union of them is not a rectangle. A tiling is called *generic* if no four tiles meet in a point. A tilling is *trivial* if it has only one tile. A tile r in a generic tiling of a square is called a *spiral* if it is contained in the interior of the square and for each edge e of r there is a tile s adjacent to r such that the straight line containing e intersects the interior of s. We show that a nontrivial generic irreducible tiling of a square has a spiral.

1 Introduction

Let R^2 be the Euclidean plane. A subset r of R^2 is a *upright rectangle* if $r = [a, b] \times [c, d]$ for some real numbers $a < b$ and $c < d$ where $[p, q]$ denotes the closed interval with end points p and q. Then the boundary ∂r of r is defined by $\partial r = \{a, b\} \times [c, d] \cup [a, b] \times \{c, d\}$. The interior $\mathrm{int} r$ of r is defined by $\mathrm{int} r = r - \partial r$. The points $(a, c), (a, d), (b, c), (b, d)$ are called the *corners* of r. Each of $\{a\} \times [c, d]$, $\{b\} \times [c, d]$, $[a, b] \times \{c\}$ and $[a, b] \times \{d\}$ is called an *edge* of r. All rectangles in this paper are upright and in the following we omit the adjective upright. Throughout this paper S denotes a rectangle, and $X = S$ or $X = R^2$. Let $\mathcal{R}$ be a set of rectangles. We say that $\mathcal{R}$ is a *tiling of* X if the following conditions hold:

(1) $X = \bigcup_{r \in \mathcal{R}} r$,

(2) if $r, s \in \mathcal{R}$ and $r \neq s$ then $r \cap s = \partial r \cap \partial s$,

(3) for any $x \in X$ there is a neighbourhood N of x in X such that only finitely many elements of $\mathcal{R}$ have non-empty intersection with N.

Then we call each element in $\mathcal{R}$ a *tile*. If $\mathcal{R}$ is a tiling of S then, by the compactness of S, we have that $\mathcal{R}$ is a finite set.

By a *line* we mean a straight line in R^2. By a *half line* we mean a straight half line in R^2. By a *line segment* we mean a straight line segment in R^2. Note that a line has no end points and separates R^2 into two regions. A half line has just one end point and a line segment has just two end points. Note that if a line L is oriented then the left-hand side of L and the right-hand side of L are well-defined.

Let $\mathcal{R}$ be a tiling of X, r a tile in $\mathcal{R}$ and x a corner of r. Let $\deg(x, \mathcal{R})$ be the number of tiles in $\mathcal{R}$ containing x. Then we have that $\deg(x, \mathcal{R}) = 1, 2, 3$ or 4 and the first two cases occur in the case that X is a rectangle and $x \in \partial X$. We call $\deg(x, \mathcal{R})$ the *degree* of x in $\mathcal{R}$. Suppose that x is a corner of r and $\deg(x, \mathcal{R}) = 3$. Let s and t be the other tiles in $\mathcal{R}$ containing x. Let L be the line containing the line segment $s \cap t$. We orient L by the direction from x to the other end point of $s \cap t$. We say that a corner x of r with $\deg(x, \mathcal{R}) = 3$ is a *right* (resp. *left*) *type corner of* r if r is contained in the left-hand (resp. right-hand) side of L. We say that a tile r in $\mathcal{R}$ is a *right* (resp. *left*) *spiral* if every corner of r is a right (resp. left) type corner. We say that a tile r in $\mathcal{R}$ is a *spiral* if it is a right spiral or a left spiral.

We say that a tile r in $\mathcal{R}$ is a *right* (resp. *left*) *pseudo spiral* if r is a right (resp. left) spiral, or one of the corners of r has degree four and the other corners are right (resp.

left) type corners of r. We say that a tile r in $\mathcal{R}$ is a *pseudo spiral* if it is a right pseudo spiral or a left pseudo spiral.

We say that a tiling $\mathcal{R}$ of X is *irreducible* if for any tiles r and s in $\mathcal{R}$ with $r \neq s$ the union $r \cup s$ is not a rectangle. We say that a tiling $\mathcal{R}$ of X is *strongly irreducible* if it is irreducible, and for any proper subset $\mathcal{Q}$ of $\mathcal{R}$ which contains at least two tiles the union $\bigcup_{r \in \mathcal{Q}} r$ is not a rectangle.

A tiling $\mathcal{R}$ is *nontrivial* if $\mathcal{R}$ contains at least two tiles. A tiling $\mathcal{R}$ is *generic* if every corner of every tile in $\mathcal{R}$ has degree less than 4.

Theorem 1. *Let $\mathcal{R}$ be a nontrivial irreducible tiling of a rectangle. Then $\mathcal{R}$ contains a pseudo spiral.*

By definition a pseudo spiral in a generic tiling is a spiral. Therefore we immediately have the following corollary.

Corollary 2. *Let $\mathcal{R}$ be a nontrivial generic irreducible tiling of a rectangle. Then $\mathcal{R}$ contains a spiral.*

We say that a tiling $\mathcal{R}$ of R^2 is *proper* if there is a real number $\varepsilon > 0$ such that any edge of any tile in $\mathcal{R}$ has the length greater than ε.

Theorem 3. *Let $\mathcal{R}$ be a proper irreducible tiling of R^2. Then $\mathcal{R}$ contains a pseudo spiral.*

Corollary 4. *Let $\mathcal{R}$ be a proper generic irreducible tiling of R^2. Then $\mathcal{R}$ contains a spiral.*

Example 5. (1) Generic irreducible tilings containing only one left spiral are illustrated in Figure 1 (a) and (b). Note that (a) is not strongly irreducible while (b) is strongly irreducible.

(2) There is a generic strongly irreducible tiling of R^2 whose tiles are all spirals as illustrated in Figure 1 (c).

(3) There is a generic strongly irreducible tiling of R^2 without spirals as illustrated in Figure 1 (d). Note that it is not proper.

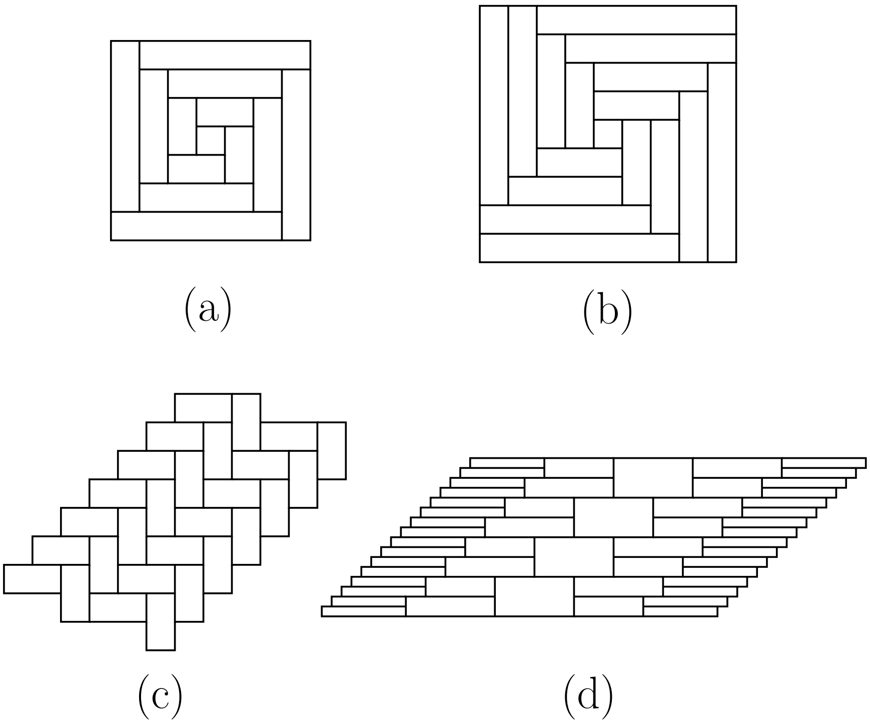

Figure 1

Let $\mathcal{R}$ be a tiling of a rectangle S. Suppose that there are tiles r and s in $\mathcal{R}$ with $r \neq s$ such that $r \cup s$ is a rectangle. Let $\mathcal{R}'$ be a tiling of S defined by $\mathcal{R}' = (\mathcal{R} - \{r, s\}) \cup \{r \cup s\}$. Then we say that $\mathcal{R}'$ is a *fusion* of $\mathcal{R}$ and $\mathcal{R}$ is a *fission* of $\mathcal{R}'$. Note that if $\mathcal{R}$ is generic then $\mathcal{R}'$ is also generic. Let $\mathcal{R}$ and $\mathcal{T}$ be tilings of S. We say that $\mathcal{T}$ is a *minor* of $\mathcal{R}$ if there is a finite sequence $\mathcal{R}_1 = \mathcal{R}, \mathcal{R}_2, \cdots, \mathcal{R}_n = \mathcal{T}$ of tilings of S such that for each $i \in \{1, 2, \cdots, n-1\}$ $\mathcal{R}_{i+1}$ is a fusion of $\mathcal{R}_i$. This relation is clearly a partial ordering on the set of the tilings of S. In order to extend this definition to the set of the tilings of R^2 we introduce the following concepts. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be tilings of X. We say that $\mathcal{R}_1$ is a *subdivision* of $\mathcal{R}_2$ if for each tile r in $\mathcal{R}_2$ there is a subset $\mathcal{Q}$ of $\mathcal{R}_1$ such that $r = \bigcup_{s \in \mathcal{Q}} s$. Then we say that $\mathcal{R}_2$ is a *weak minor* of $\mathcal{R}_1$. It is clear that this is a partial ordering on the set of the tilings of X. Suppose that $\mathcal{R}_1$ and $\mathcal{R}_2$ are tilings of R^2 and $\mathcal{R}_2$ is a weak minor of $\mathcal{R}_1$. Suppose that for each tile r in $\mathcal{R}_2$ the trivial tiling of r is a minor of the tiling $\mathcal{Q}$ of a rectangle r where $\mathcal{Q}$ is the subset of $\mathcal{R}_1$ with $r = \bigcup_{s \in \mathcal{Q}} s$. Then we say that $\mathcal{R}_2$ is a *minor* of $\mathcal{R}_1$. It is clear that this is also a partial ordering on the set of the tilings of R^2. Suppose that $\mathcal{R}_1$ and $\mathcal{R}_2$ are tilings of X. We note that if $\mathcal{R}_2$ is a minor of $\mathcal{R}_1$ then $\mathcal{R}_2$ is a weak minor of $\mathcal{R}_1$. We also note that any weak minor of a generic tiling of X is generic.

Now we are interested in the structures of these partial orderings and have the following results.

Theorem 6. *Let $\mathcal{R}$ be a generic tiling of a rectangle S.*

(1) *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be minors of $\mathcal{R}$. Then there is a minor $\mathcal{R}_3$ of $\mathcal{R}$ such that $\mathcal{R}_3$ is a common minor of $\mathcal{R}_1$ and $\mathcal{R}_2$.*

(2) *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be irreducible minors of $\mathcal{R}$. Then $\mathcal{R}_1 = \mathcal{R}_2$.*

Theorem 7. *Let $\mathcal{R}$ be a generic tiling of R^2.*

(1) *Suppose that there is an irreducible minor $\mathcal{R}_0$ of $\mathcal{R}$. Then for any minor $\mathcal{R}_1$ of $\mathcal{R}$, $\mathcal{R}_0$ is a minor of $\mathcal{R}_1$.*

(2) *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be irreducible minors of $\mathcal{R}$. Then $\mathcal{R}_1 = \mathcal{R}_2$.*

Theorem 8. *Let $\mathcal{R}$ be a generic tiling of a rectangle S or R^2.*

(1) *Suppose that there is a strongly irreducible weak minor $\mathcal{R}_0$ of $\mathcal{R}$. Then for any nontrivial weak minor $\mathcal{R}_1$ of $\mathcal{R}$, $\mathcal{R}_0$ is a weak minor of $\mathcal{R}_1$.*

(2) *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be nontrivial strongly irreducible weak minors of $\mathcal{R}$. Then $\mathcal{R}_1 = \mathcal{R}_2$.*

Remark 9. (1) If a tiling is not generic then there may be many strongly irreducible minors. For example the tiling of a square illustrated in Figure 2 (a) has three strongly irreducible minors as illustrated in Figure 2 (b).

(2) There is a generic tiling of R^2 that has no irreducible minors. An example $\mathcal{R}$ is illustrated in Figure 2 (c). Note that the minors $\mathcal{R}_1$ and $\mathcal{R}_2$ of $\mathcal{R}$ illustrated in Figure 2 (d) have no common minors.

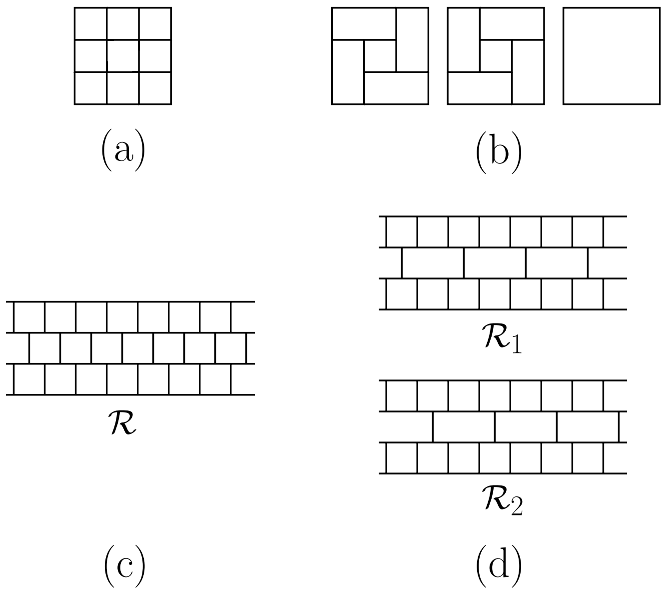

Figure 2

Let $\mathcal{R}$ be a generic tiling of a rectangle S and $\mathcal{Q}$ a proper subset of $\mathcal{R}$. We say that $\mathcal{Q}$ is a *spiral-subdivision* if $\bigcup_{r\in\mathcal{Q}} r$ is a rectangle and it is a spiral in the tiling $\mathcal{R}' = (\mathcal{R} - \mathcal{Q}) \cup \{\bigcup_{r\in\mathcal{Q}} r\}$ of S. We note that $\mathcal{Q}$ may contain only one tile.

It is clear that a tiling of a rectangle has a minor that is irreducible. By Theorem 6 such an irreducible minor is unique. Then as a consequence of Corollary 2 we have the following theorem.

Theorem 10. *Let $\mathcal{R}$ be a generic tiling of a rectangle S. Then the irreducible minor of $\mathcal{R}$ is not the trivial tiling of S if and only if $\mathcal{R}$ contains a spiral-subdivision.*

2 Proofs

Let $\mathcal{R}$ be a tiling of X where X is a rectangle S or R^2. A line segment L in X is called a *pre-road* of $\mathcal{R}$ if L is a union of some edges of some tiles in $\mathcal{R}$. A pre-road L of $\mathcal{R}$ is called a *road* of $\mathcal{R}$ if $X = R^2$, or $X = S$ and L is not contained in ∂S. A *route* $\tilde{L}$ is a road L together with a fixed orientation. The end points of a route $\tilde{L}$ consists of a *starting point* and a *terminal point* so that the orientation of $\tilde{L}$ coincides with the one from the starting point to the terminal point. A route $\tilde{L}$ is *extendable* if there is a route $\tilde{M}$ of $\mathcal{R}$

such that L is a proper subset of M and $\tilde{L}$ and $\tilde{M}$ have the same starting point. Then we say that $\tilde{M}$ is an *extension* of $\tilde{L}$. A route that is not extendable is called *maximal.* A sequence $(\tilde{L}_1, \tilde{L}_2, \cdots, \tilde{L}_n)$ of routes in $\mathcal{R}$ is called a *right turning walk* (resp. *left turning walk*) of $\mathcal{R}$ if for each $i \in \{1, 2, \cdots, n-1\}$, $\tilde{L}_i$ is a maximal route, the terminal point of $\tilde{L}_i$ is the starting point of $\tilde{L}_{i+1}$ and L_{i+1} lies in the right-hand side of $\tilde{L}_i$ (resp. in the left-hand side of $\tilde{L}_i$). Note that $\tilde{L}_n$ is not necessarily maximal.

Lemma 11. *Let $\mathcal{R}$ be a strongly irreducible tiling of a rectangle or R^2. Suppose that there is a right (resp. left) turning walk $(\tilde{L}_1, \tilde{L}_2, \tilde{L}_3, \tilde{L}_4)$ such that the line containing L_4 intersects L_1 at a point that is not the starting point of $\tilde{L}_1$. Then $\mathcal{R}$ contains a right (resp. left) pseudo spiral.*

Proof. By extending $\tilde{L}_4$ if necessary we may suppose without loss of generality that either L_4 intersects L_1 or $\tilde{L}_4$ is a maximal route. First suppose that L_4 intersects L_1. Note that then the point of intersection is not the starting point of $\tilde{L}_1$. Let r_1 be the rectangle bounded by the line segments L_1, L_2, L_3 and L_4. Since $\mathcal{R}$ is strongly irreducible we have that r_1 is a tile in $\mathcal{R}$. Then we have that r_1 is a right (reps. left) pseudo spiral. Next suppose that L_4 does not intersect L_1. Then we consider a right (resp. left) turning walk $(\tilde{L}_2, \tilde{L}_3, \tilde{L}_4, \tilde{L}_5)$. By extending $\tilde{L}_5$ if necessary we may suppose without loss of generality that either L_5 intersects L_2 or $\tilde{L}_5$ is a maximal route. Suppose that L_5 intersects L_2. Note that then the point of intersection is not the starting point of $\tilde{L}_2$. Let r_2 be the rectangle bounded by the line segments L_2, L_3, L_4 and L_5. Since $\mathcal{R}$ is strongly irreducible we have that r_2 is a tile in $\mathcal{R}$. Then we have that r_2 is a right (reps. left) pseudo spiral. Suppose that L_5 does not intersect L_2. Then we consider a right (resp. left) turning walk $(\tilde{L}_3, \tilde{L}_4, \tilde{L}_5, \tilde{L}_6)$ and repeat the discussion above. By repeating the arguments we finally find a right (reps. left) pseudo spiral. See Figure 3. □

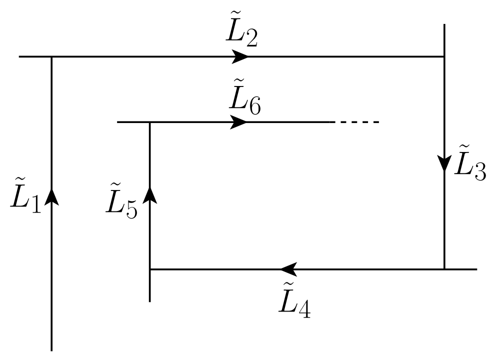

Figure 3

Proof of Theorem 1. Let $\mathcal{R}$ be a nontrivial irreducible tiling of a rectangle S. We give the proof by the induction on the number k of the tiles in $\mathcal{R}$. Every nontrivial tiling contains at least two tiles. Thus Theorem 1 is true for $k = 1$. Now suppose that Theorem 1 is true for all $k \leq l$. Suppose that $\mathcal{R}$ contains just $l+1$ tiles. Suppose that $\mathcal{R}$ is not strongly irreducible. Then there is a rectangle $S' \subset S$ and a nontrivial irreducible tiling $\mathcal{R}' \subset \mathcal{R}$ of S' such that the number of tiles in $\mathcal{R}'$ is less than or equal to l. Then by the induction hypothesis we have that $\mathcal{R}'$ has a pseudo spiral. Therefore $\mathcal{R}$ also has a pseudo spiral.

Next we consider the case that $\mathcal{R}$ is strongly irreducible. Let x be a corner of some tile in $\mathcal{R}$ with $\deg(x, \mathcal{R}) = 2$. Note that x is a point on ∂S. Let $\tilde{L}_1$ be the maximal road of $\mathcal{R}$ starting from x. Since $\mathcal{R}$ is strongly irreducible we have that the terminal point of $\tilde{L}_1$ is not on ∂S. Then we have a left turning walk $(\tilde{L}_1, \tilde{L}_2)$. By extending $\tilde{L}_2$ if necessary we may suppose without loss of generality that $\tilde{L}_2$ is maximal.

Case 1. The terminal point of $\tilde{L}_2$ is on ∂S.
We treat Case 1 later.

Case 2. The terminal point of $\tilde{L}_2$ is not on ∂S.
Let $(\tilde{L}_1, \tilde{L}_2, \tilde{L}_3)$ be a left turning walk. We may suppose that $\tilde{L}_3$ is maximal.

Case 2.1. The terminal point of $\tilde{L}_3$ is on ∂S.
We treat Case 2.1 later.

Case 2.2. The terminal point of $\tilde{L}_3$ is not on ∂S.
Then we have a left turning walk $(\tilde{L}_1, \tilde{L}_2, \tilde{L}_3, \tilde{L}_4)$ and by Lemma 11 we have a left pseudo spiral.

Case 1 and Case 2.1
In these cases we have, by the strong irreducibility of $\mathcal{R}$, that there is a right turning walk $(\tilde{L}_1, \tilde{M}_2, \tilde{M}_3, \tilde{M}_4)$. Then by Lemma 11 we have a right pseudo spiral. This completes the proof. $\square$

Lemma 12. *Let $\mathcal{R}$ be a strongly irreducible proper tiling of R^2. Let L be a road of $\mathcal{R}$. Let x_1 and x_2 be the end points of L. Let H_1 and H_2 be mutually disjoint and mutually parallel half lines starting from x_1 and x_2 respectively so that $L \cup H_1 \cup H_2$ bounds a convex region in R^2. Suppose that both of H_1 and H_2 are union of some edges of some tiles in $\mathcal{R}$. Then there is a pseudo spiral of $\mathcal{R}$ in the convex region.*

Proof. Let A be the closure of the convex region bounded by $L \cup H_1 \cup H_2$. Since $\mathcal{R}$ is proper there are only finitely many half lines contained in A each of which is a union of some edges of some tiles in $\mathcal{R}$. Therefore, by re-choosing L, H_1 and H_2 if necessary, we may suppose that there are no such road L' and half lines H'_1 and H'_2 in A with the distance between H'_1 and H'_2 closer than that of H_1 and H_2.

We give orientations to H_1 and H_2 such that x_1 and x_2 are the starting points of them respectively. We may suppose without loss of generality that H_2 lies in the left-hand side of H_1. Let x be a corner of some tile in $\mathcal{R}$ with $x \in H_1$ and $x \neq x_1$. Let $\tilde{M}_1$ be a route of $\mathcal{R}$ starting from x toward the half line H_2. By extending $\tilde{M}_1$ if necessary we may suppose that M_1 intersects H_2 or $\tilde{M}_1$ is a maximal route of $\mathcal{R}$. First suppose that M_1 intersects H_2. Then we have a tile in $\mathcal{R}$ bounded by L, H_1, H_2 and M_1. Then we choose x' on H_1 that is not on the tile, and consider a route $\tilde{M}'_1$ starting from x' toward H_2. Then by the strong irreducibility of $\mathcal{R}$ we may suppose that $\tilde{M}'_1$ is maximal and M'_1 does not intersect H_2. Thus we may suppose without loss of generality that M_1 does not intersect H_2. Then by the assumption we have a right turning walk ($\tilde{M}_1$, $\tilde{M}_2$, $\tilde{M}_3$). By extending $\tilde{M}_3$ if necessary we may suppose without loss of generality that M_3 intersects H_1 or $\tilde{M}_3$ is a maximal route. Suppose that M_3 does not intersect H_1. Then we have a right turning walk ($\tilde{M}_1$, $\tilde{M}_2$, $\tilde{M}_3$, $\tilde{M}_4$) and have a right pseudo spiral by Lemma 11. Suppose that M_3 intersects H_1. Then we have a tile in $\mathcal{R}$ bounded by H_1, M_1, M_2 and M_3. Then by the strong irreducibility of $\mathcal{R}$ we have a left turning walk ($\tilde{M}_1$, $\tilde{N}_2$, $\tilde{N}_3$, $\tilde{N}_4$) and have a left pseudo spiral by Lemma 11. $\square$

Lemma 13. *Let $\mathcal{R}$ be a strongly irreducible tiling of R^2. Let e be an edge of a tile r in $\mathcal{R}$. Let x_1 and x_2 be the end points of e. Suppose that there is a maximal route $\tilde{L}_1$ of $\mathcal{R}$ starting from x_1 such that L_1 is perpendicular to e and $r \cap L_1 = \{x_1\}$. Suppose that there is a route $\tilde{M}_1$ of $\mathcal{R}$ starting from x_2 such that M_1 is perpendicular to e and $r \cap M_1 = \{x_2\}$. Then there is a pseudo spiral of $\mathcal{R}$.*

Proof. We consider the case that M_1 lies in the right-hand side of the line containing L_1 oriented by that of $\tilde{L}_1$. The other case is entirely analogous and we omit it. Let ($\tilde{L}_1$, $\tilde{L}_2$) be a right turning walk of $\mathcal{R}$. First suppose that $\tilde{L}_2$ is maximal and the length of L_2 is less than the length of e. Then by considering a right turning walk ($\tilde{L}_1$, $\tilde{L}_2$, $\tilde{L}_3$, $\tilde{L}_4$) and by Lemma 11 we have a right pseudo spiral. Next suppose that $\tilde{L}_2$ is maximal and the length of L_2 is equal to the length of e. Then by the strong irreducibility of $\mathcal{R}$ we have a right turning walk ($\tilde{L}_1$, $\tilde{L}_2$, $\tilde{L}_3$, $\tilde{L}_4$). Then by Lemma 11 we have a right pseudo spiral. Finally suppose the length of $\tilde{L}_2$ is greater than the length of e. Then by the strong irreducibility of $\mathcal{R}$ we have a left turning walk ($\tilde{M}_1$, $\tilde{M}_2$, $\tilde{M}_3$, $\tilde{M}_4$) and by Lemma

11 we have a left pseudo spiral. □

Proof of Theorem 3. First suppose that $\mathcal{R}$ is not strongly irreducible. Then there is a rectangle S and a subset $\mathcal{R}'$ of $\mathcal{R}$ such that $\mathcal{R}'$ is a nontrivial irreducible tiling of S. Then by Theorem 1 there is a pseudo spiral in $\mathcal{R}'$. Then it is also a pseudo spiral in $\mathcal{R}$.

Next suppose that $\mathcal{R}$ is strongly irreducible. Let r be a tile in $\mathcal{R}$. If r is a pseudo spiral then we have the result. Suppose that r is not a pseudo spiral. Then there is an edge e of r with the following properties. Let x_1 and x_2 be the end points of e. Then there are routes $\tilde{L}_1$ and $\tilde{L}_2$ starting from x_1 and x_2 respectively such that they are mutually parallel, $r \cap L_1 = \{x_1\}$ and $r \cap L_2 = \{x_2\}$. Suppose that at least one of them is maximal. Then by Lemma 13 we have a pseudo spiral. Therefore it is sufficient to consider the situation that both of them are endlessly extendable. Then we have a situation in Lemma 12 and have a pseudo spiral. □

Lemma 14. *Let $\mathcal{R}$ be a generic tiling of a rectangle S or R^2. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be minors of $\mathcal{R}$ such that $\mathcal{R}_1$ is irreducible. Then $\mathcal{R}_1$ is a weak minor of $\mathcal{R}_2$.*

Proof. It is sufficient to show that each tile r in $\mathcal{R}_2$ is contained in some tile in $\mathcal{R}_1$. Let $\mathcal{Q}$ be the subset of $\mathcal{R}$ with $r = \bigcup_{s \in \mathcal{Q}} s$. Then $\mathcal{Q}$ is a generic tiling of r and the trivial tiling $\mathcal{T}$ of r is a minor of $\mathcal{Q}$. Therefore there is a sequence of tilings $\mathcal{Q} = \mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_n = \mathcal{T}$ such that $\mathcal{P}_{k+1}$ is a fusion of $\mathcal{P}_k$ for each $k \in \{1, 2, \cdots, n-1\}$. Note that each tile in $\mathcal{P}_1$ is contained in some tile in $\mathcal{R}_1$. Suppose inductively that each tile in $\mathcal{P}_k$ is contained in some tile in $\mathcal{R}_1$. Let r_1 and r_2 be the tiles in $\mathcal{P}_k$ such that $r_1 \cup r_2 \in \mathcal{P}_{k+1}$. If $r_1 \cup r_2$ is contained in a tile in $\mathcal{R}_1$ then we have that each tile in $\mathcal{P}_{k+1}$ is contained in some tile in $\mathcal{R}_1$. If r_1 and r_2 are contained in different tiles in $\mathcal{R}_1$ then we have that either $\mathcal{R}$ is not generic or $\mathcal{R}_1$ is not irreducible. Thus we have that this case does not occur. Therefore we have that each tile in $\mathcal{P}_{k+1}$ is contained in some tile in $\mathcal{R}_1$. Therefore we have that r is contained in some tile in $\mathcal{R}_1$. This completes the proof. □

Proof of Theorem 6. We first show that (2) implies (1). Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be minors of $\mathcal{R}$. It is clear that there are irreducible minors $\mathcal{R}_3$ and $\mathcal{R}_4$ of $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively. Then $\mathcal{R}_3$ and $\mathcal{R}_4$ are irreducible minors of $\mathcal{R}$. Then by (2) we have $\mathcal{R}_3 = \mathcal{R}_4$. Then $\mathcal{R}_3 = \mathcal{R}_4$ is a common minor of $\mathcal{R}_1$ and $\mathcal{R}_2$.

Therefore it is sufficient to show (2). Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be irreducible minors of $\mathcal{R}$. Then by Lemma 14 we have that $\mathcal{R}_1$ is a weak minor of $\mathcal{R}_2$ and $\mathcal{R}_2$ is a weak minor of $\mathcal{R}_1$. Therefore we have $\mathcal{R}_1 = \mathcal{R}_2$. □

Proof of Theorem 7. We first show that (1) implies (2). Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be irreducible minors of $\mathcal{R}$. Then by (1) we have that $\mathcal{R}_1$ is a minor of $\mathcal{R}_2$ and $\mathcal{R}_2$ is a minor of $\mathcal{R}_1$. Therefore we have $\mathcal{R}_1 = \mathcal{R}_2$.

Therefore it is sufficient to show (1). Suppose that $\mathcal{R}_0$ is an irreducible minor of $\mathcal{R}$ and $\mathcal{R}_1$ is a minor of $\mathcal{R}$. Then by Lemma 14 we have that $\mathcal{R}_0$ is a weak minor of $\mathcal{R}_1$. Let r be a tile in $\mathcal{R}_0$. Let $\mathcal{T}$ be a subset of $\mathcal{R}$ such that $r = \bigcup_{t\in\mathcal{T}} t$. Let $\mathcal{T}_1$ be a subset of $\mathcal{R}_1$ such that $r = \bigcup_{t\in\mathcal{T}_1} t$. Then it is clear that $\mathcal{T}_1$ is a minor of $\mathcal{T}$. Since the trivial tiling of r is an irreducible minor of $\mathcal{T}$ we have by Theorem 6 that the trivial tiling of r is also a minor of $\mathcal{T}_1$. Thus we have that $\mathcal{R}_0$ is a minor of $\mathcal{R}_1$. $\square$

Proof of Theorem 8. We first show that (1) implies (2). Suppose that $\mathcal{R}_1$ and $\mathcal{R}_2$ are nontrivial strongly irreducible weak minors of $\mathcal{R}$. Then by (1) we have that $\mathcal{R}_1$ is a weak minor of $\mathcal{R}_2$ and $\mathcal{R}_2$ is a weak minor of $\mathcal{R}_1$. Therefore we have $\mathcal{R}_1 = \mathcal{R}_2$.

Therefore it is sufficient to show (1). We will show that for each tile r_1 in $\mathcal{R}_1$ there is a tile r_0 in $\mathcal{R}_0$ such that $r_1 \subset r_0$. Suppose contrary that there is a tile r_1 in $\mathcal{R}_1$ such that for any tile r in $\mathcal{R}_0$ r_1 is not contained in r. Suppose in addition that for each tile r_0 in $\mathcal{R}_0$ with $\mathrm{int}r_1 \cap \mathrm{int}r_0 \neq \emptyset$, r_0 is contained in r_1. Then we have that $\mathcal{R}_0$ is not strongly irreducible. Therefore we have that there is a tile r_0 in $\mathcal{R}_0$ with $\mathrm{int}r_1 \cap \mathrm{int}r_0 \neq \emptyset$ such that r_0 is not contained in r_1. Since $\mathcal{R}$ is generic we have that both $r_1 \cap r_0$ and $r_1 \cup r_0$ are rectangles. Suppose that for each tile s_0 in $\mathcal{R}_0$ with $s_0 \neq r_0$ and $\mathrm{int}r_1 \cap \mathrm{int}s_0 \neq \emptyset$, s_0 is contained in r_1. Then we have that $\mathcal{R}_0$ is not strongly irreducible. Therefore there is a tile s_0 in $\mathcal{R}_0$ with $s_0 \neq r_0$ and $\mathrm{int}r_1 \cap \mathrm{int}s_0 \neq \emptyset$ such that s_0 is not contained in r_1. Since $\mathcal{R}$ is generic we have that both $r_1 \cap s_0$ and $r_1 \cup s_0$ are rectangles. By the strong irreducibility of $\mathcal{R}_0$ we have that r_0 and s_0 are disjoint. Then, since $\mathcal{R}$ is generic, we have that for each tile t_0 in $\mathcal{R}_0$ with $t_0 \neq r_0, s_0$ and $\mathrm{int}r_1 \cap \mathrm{int}t_0 \neq \emptyset$, t_0 is contained in r_1. Then we have that $\mathcal{R}_0$ is not strongly irreducible. This is a contradiction. Therefore we have that for each tile r_1 in $\mathcal{R}_1$ there is a tile r_0 in $\mathcal{R}_0$ such that $r_1 \subset r_0$. $\square$

Proof of Theorem 10. First suppose that $\mathcal{R}$ contains a spiral-subdivision $\mathcal{Q}$. Let $s = \bigcup_{r\in\mathcal{Q}} r$. Suppose that $\mathcal{R}_1$ is a minor of $\mathcal{R}$. Since $\mathcal{R}$ is generic we have, by an inductive argument, that there is a subset $\mathcal{Q}_1$ of $\mathcal{R}_1$ such that $s = \bigcup_{r\in\mathcal{Q}_1} r$. Therefore $\mathcal{R}_1$ is not trivial. Thus the trivial tiling of S is not a minor of $\mathcal{R}$. Next suppose that the irreducible minor of $\mathcal{R}$ is not the trivial tiling of S. Then by Corollary 2 we have that the irreducible minor of $\mathcal{R}$ contains a spiral. Then we have that $\mathcal{R}$ contains a spiral-subdivision. $\square$