多重ゼータ値と有限オートマトン

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2004年3月28日

多重ゼータ値(MZV)

$$\zeta(k_1, k_2, \ldots, k_n) = \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}$$

$k_1, k_2, \ldots, k_n \in \mathbb{Z}_{\geq 1}$, $(k_1 \geq 2)$.
$k = k_1 + \cdots + k_n$: 重み(weight)
n : 深さ(depth)

Zagier-Broadhurst の公式

$$\zeta(\{3,1\}_n) = \zeta(\underbrace{3, 1, \ldots, 3, 1}_{n\,\text{times}}) = \frac{2\pi^{4n}}{(4n+2)!}.$$

ZB公式のZagierによる証明

$$\sum_{n=0}^{\infty} Li_{\{3,1\}_n}(x)\, t^{4n} = F\Bigl(\frac{t}{1+i}, \frac{-t}{1+i}; 1; x\Bigr)\, F\Bigl(\frac{t}{1-i}, \frac{-t}{1-i}; 1; x\Bigr)$$

$$F(a, -a; 1; 1) = \frac{1}{\Gamma(1-a)\Gamma(1+a)} = \frac{\sin \pi a}{\pi a}$$

$$F\Bigl(\frac{t}{1+i}, \frac{-t}{1+i}; 1; 1\Bigr)\, F\Bigl(\frac{t}{1-i}, \frac{-t}{1-i}; 1; 1\Bigr) = \sum_{n=0}^{\infty} \frac{2\pi^{4n}}{(4n+2)!}\, t^{4n}$$

シャッフル代数の導入

$\mathfrak{H} = \mathbb{Q}\langle x, y\rangle \supset \mathfrak{H}^0 = \mathfrak{H}\cdot 1 + x\mathfrak{H}y,\ z_k = x^{k-1}y \in \mathfrak{H}^0$

$Z : \mathfrak{H}^0 \longrightarrow \mathbb{R}$ ($\mathbb{Q}$-線型)

$$Z(z_1 z_2 \cdots z_n) = \zeta(k_1, k_2, \ldots, k_n)$$

定義　$ш : \mathfrak{H}\times\mathfrak{H} \longrightarrow \mathfrak{H}$ （シャッフル積）

(i) $w \,ш\, 1 = 1 \,ш\, w = w$ $(w \in \mathfrak{H})$

(ii) $u_1,\ u_2 = x, y$　とする．任意の語$w_1, w_2 \in \mathfrak{H}$に対して

$$u_1 w_1 \,ш\, u_2 w_2 = u_1(w_1 \,ш\, u_2 w_2) + u_2(u_1 w_1 \,ш\, w_2)$$

$(\mathfrak{H},\ ш)$　**シャッフル代数**という．可換代数である．

準同型性　　$Z(w_1 \,ш\, w_2) = Z(w_1)Z(w_2)$ $(w_1, w_2 \in \mathfrak{H}^0)$

Waldschmidtのアイデア

$w \in x\mathfrak{H} + y\mathfrak{H}$ に対して

$$w^\star = \sum_{n=0}^{\infty} w^n \in \widehat{\mathfrak{H}} = \mathbb{Q}\langle\langle x, y\rangle\rangle \quad \text{(Kleene 閉包)}$$

Waldschmidtの公式 ： $(xy)^\star ш (-xy)^\star = (-4x^2y^2)^\star$

$$\sum_{p=0}^{2n} (-1)^{2n-p} \zeta(\{2\}_p)\zeta(\{2\}_{2n-p}) = (-4)^n \zeta(\{3,1\}_n)$$

$$LHS = (-1)^n \zeta(\{4\}_n) = \frac{2 \cdot (-4)^n \pi^{4n}}{(4n+2)!}$$

有限オートマトンの定義

1. 有限オートマトン$(Q, \Sigma, \delta, q_1, F)$ とは
   (a) $Q = \{q_1, q_2, \ldots, q_m\}$ 状態の集合;
   (b) Σ アルファベット（x, yの線型結合）の有限集合;
   (c) $\delta : Q \times \Sigma \longrightarrow Q$ 遷移写像;
   (d) q_1 初期状態;
   (e) $F \subset Q$ 最終状態.

2. $\delta(\cdots(\delta(q_1, u_1)\cdots u_n) = q_i \in F$ が成立するとき，
   語 $w = u_1 \cdots u_n$ は受理されるという．また，この語の長さはnであるという.

オートマトン$M=(Q,\Sigma,\delta,q_1,F)$に対して$\widehat{\mathfrak{H}}$の元 $w=w(M)$が

$$w(M)=\sum(M\text{によって受理される語})\times\sharp(\text{語を受理する道})$$

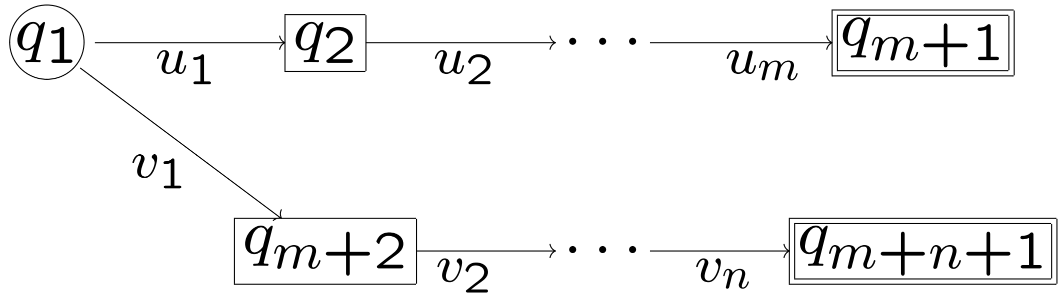
で定まる.

オートマトンの例

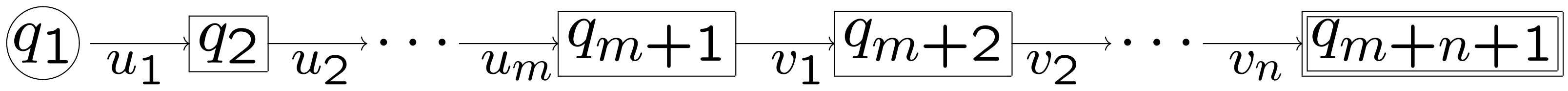
$$w_1=u_1\cdots u_m : \qquad q_1 \xrightarrow[u_1]{} \boxed{q_2} \xrightarrow[u_2]{} \cdots \xrightarrow[u_m]{} \boxed{\boxed{q_{m+1}}}\ ,$$

$$w_2=v_1\cdots v_n : \qquad q_1 \xrightarrow[v_1]{} \boxed{q_2} \xrightarrow[v_2]{} \cdots \xrightarrow[v_n]{} \boxed{\boxed{q_{n+1}}}\ .$$

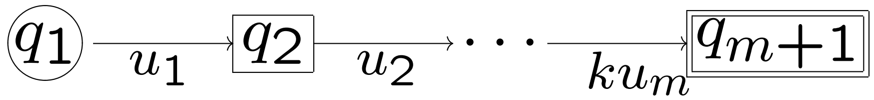
$w_1 + w_2$:



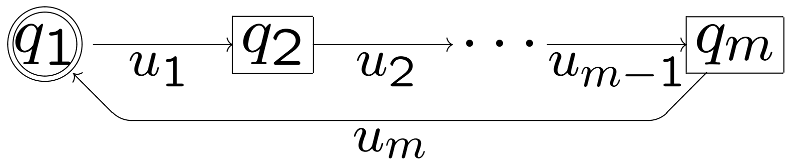
$w_1 w_2$:



$k w_1$:

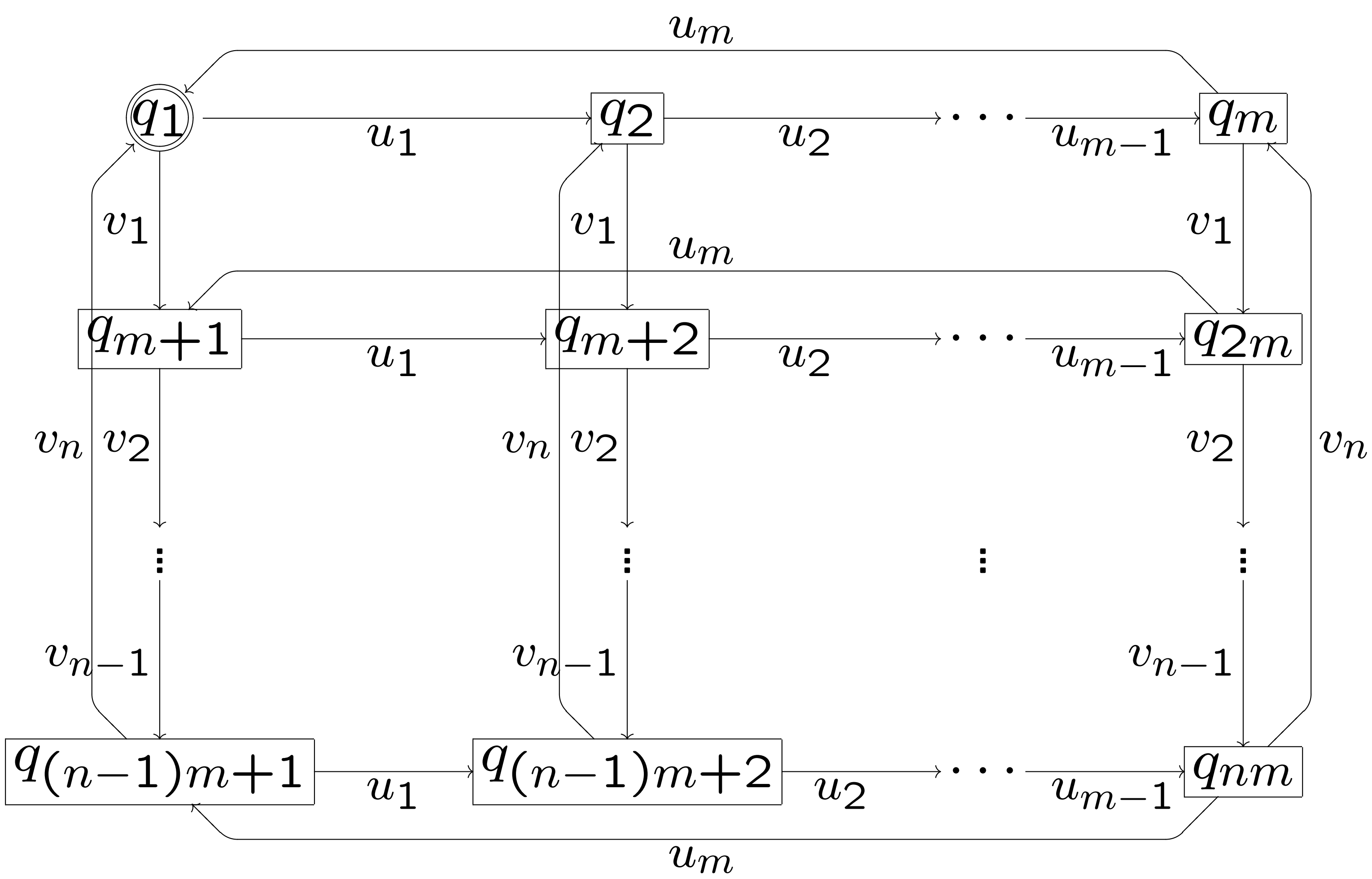


$w_1^\star$:



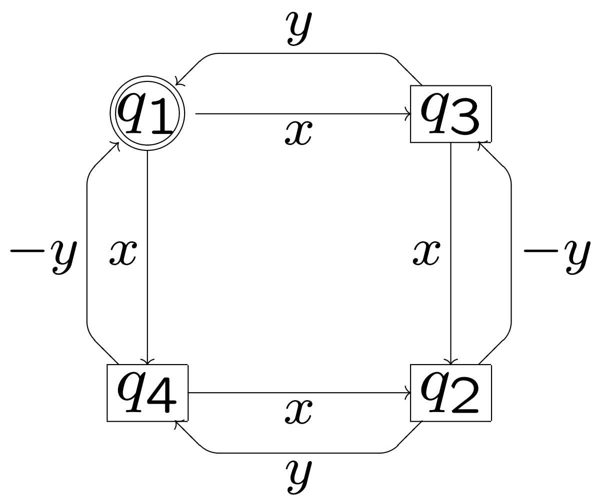
命題 1 シャッフル積$w_1 \sqcup\!\sqcup\, w_2$，$(w_1)^\star \sqcup\!\sqcup\, (w_2)^\star$は次のオートマトンにより表現される．これを**シャッフル・オートマトン**と呼ぶことにする．

$$
\begin{tikzcd}
q_1 \arrow[r, "u_1"'] \arrow[d, "v_1"'] & q_2 \arrow[r, "u_2"'] \arrow[d, "v_1"'] & \cdots \arrow[r, "u_m"'] & q_{m+1} \arrow[d, "v_1"'] \\
q_{(m+1)+1} \arrow[r, "u_1"'] \arrow[d, "v_2"'] & q_{(m+1)+2} \arrow[r, "u_2"'] \arrow[d, "v_2"'] & \cdots \arrow[r, "u_m"'] & q_{2(m+1)} \arrow[d, "v_2"'] \\
\vdots \arrow[d, "v_{n-1}"'] & \vdots \arrow[d, "v_{n-1}"'] & \vdots & \vdots \arrow[d, "v_{n-1}"'] \\
q_{(n-1)(m+1)+1} \arrow[r, "u_1"'] \arrow[d, "v_n"'] & q_{(n-1)(m+1)+2} \arrow[r, "u_2"'] \arrow[d, "v_n"'] & \cdots \arrow[r, "u_m"'] & q_{n(m+1)} \arrow[d, "v_n"'] \\
q_{n(m+1)+1} \arrow[r, "u_1"'] & q_{n(m+1)+2} \arrow[r, "u_2"'] & \cdots \arrow[r, "u_m"'] & q_{(n+1)(m+1)}
\end{tikzcd}
$$


u_m
q_1
u_1
q_2
u_2
$\cdots$
u_{m-1}
q_m
v_1
v_1
v_1
u_m
q_{m+1}
u_1
q_{m+2}
u_2
$\cdots$
u_{m-1}
q_{2m}
v_n
v_2
v_n
v_2
v_2
v_n
$\vdots$
$\vdots$
$\vdots$
$\vdots$
v_{n-1}
v_{n-1}
v_{n-1}
$q_{(n-1)m+1}$
u_1
$q_{(n-1)m+2}$
u_2
$\cdots$
u_{m-1}
q_{nm}
u_m

Waldschmidtによる$(xy)^\star \sqcup\!\sqcup (-xy)^\star = (-4x^2y^2)^\star$の証明

$(xy)^\star \sqcup\!\sqcup (-xy)^\star$のシャッフル・オートマトン$M = (Q, \Sigma, \delta, q_1, F)$:



$M_k = (Q, \Sigma, \delta, q_k, F)$とする．$S_k$を$M_k$により受理される元とする．

つぎの線型再帰方程式が成立する．

$$\begin{cases} S_1 = 1 + xS_3 + xS_4, & S_2 = -yS_3 + yS_4, \\ S_3 = yS_1 + xS_2, & S_4 = -yS_1 + xS_2. \end{cases}$$

これを解いて　$S_1 = 1 - 4x^2y^2S_1.$

$$\therefore \quad S_1 = (-4x^2y^2)^\star$$

定義　$M=(Q,\Sigma,\delta,q_1,F)$ の**隣接行列**　$A=A(M)$

$$A(M)=(a_{ij}) \quad a_{ij}=\sum_{a\in\Sigma\,;\,\delta(q_i,a)=q_j} a.$$

定理 2 $A(M)^n=(a_{ij}^{(n)})$ とすると $a_{ij}^{(n)}$ は q_i から q_j へ至る長さ n の道によって受理される語の総和．したがって,

$$w(M)=\sum_{j\,;\,q_j\in F}\left(\sum_{n=0}^{\infty} a_{1j}^{(n)}\right).$$

特に，$F=\{q_1\}$ の場合は

$$w(M)=\sum_{n=0}^{\infty} a_{11}^{(n)}.$$

$$\zeta(k_1)\zeta(k_2) = \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2).$$

調和積（級数積）$*$ が $\mathfrak{H}^1 = \mathbb{Q} \cdot 1 + \mathfrak{H}y$ 上に定義される：

(i) $w * 1 = 1 * w = w$ $(w \in \mathfrak{H}^1)$.

(ii) $z_k = x^{k-1}y$ $(k = 1, 2, \ldots)$, 語 w_1, w_2 in $\mathfrak{H}^1$ に対して

$$\begin{aligned} z_i w_1 * z_j w_2 = z_i(w_1 * z_j w_2) + z_j(z_i w_1 * w_2) \\ + z_{i+j}(w_1 * w_2). \end{aligned}$$

（準同型性） $Z(w_1 * w_2) = Z(w_1)Z(w_2)$, $(w_1,\ w_2 \in \mathfrak{H}^0)$.

命題 3 (有限複シャッフル関係式)

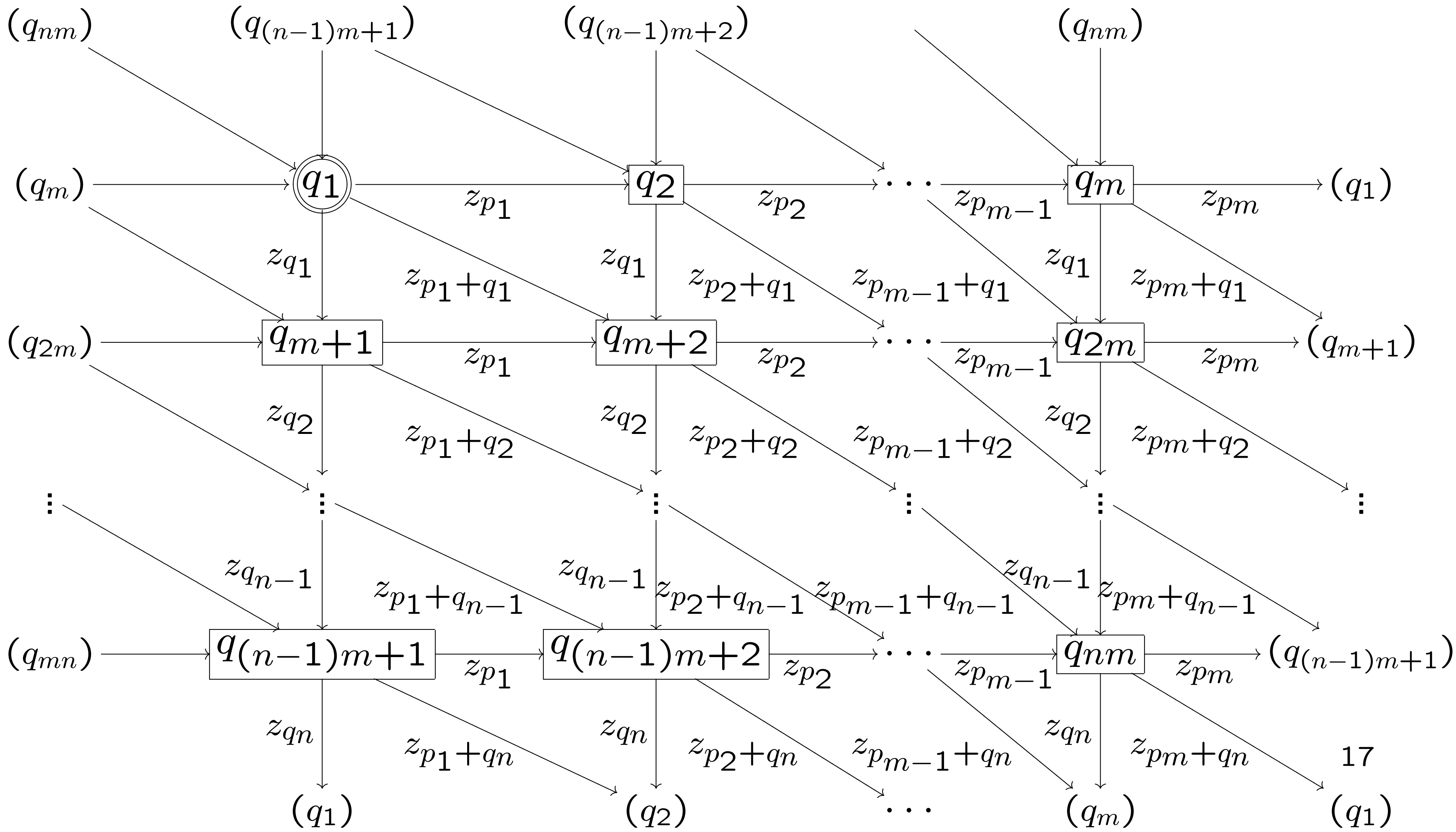
$$Z(w_1 ш w_2 - w_1 * w_2) = 0, \ (w_1, \ w_2 \in \mathfrak{H}^0)$$

定理 4 $z_k = x^{k-1}y$, ωを1の原始m乗根とする.

$$(z_k)^\star * (\omega z_k)^\star * \cdots * (\omega^{m-1} z_k)^\star = ((-1)^{m-1} z_{mk})^\star.$$

定理 5 $w_1 = z_{p_1} z_{p_2} \cdots z_{p_m}$, $w_2 = z_{q_1} z_{q_2} \cdots z_{q_n}$とする．調和積 $w_1 * w_2$, $(w_1)^\star * (w_2)^\star$を表現するオートマトンはつぎである．これを**調和オートマトン**とよぶ.

$$
\begin{tikzcd}
q_1 \arrow[r, "z_{p_1}"'] \arrow[d, "z_{q_1}"'] \arrow[dr, "z_{p_1+q_1}"'] & q_2 \arrow[r, "z_{p_2}"'] \arrow[d, "z_{q_1}"'] \arrow[dr, "z_{p_2+q_1}"'] & \cdots \arrow[r, "z_{p_m}"'] \arrow[dr, "z_{p_m+q_1}"'] & q_{m+1} \arrow[d, "z_{q_1}"'] \\
q_{(m+1)+1} \arrow[r, "z_{p_1}"'] \arrow[d, "z_{q_2}"'] \arrow[dr, "z_{p_1+q_2}"'] & q_{(m+1)+2} \arrow[r, "z_{p_2}"'] \arrow[d, "z_{q_2}"'] \arrow[dr, "z_{p_2+q_2}"'] & \cdots \arrow[r, "z_{p_m}"'] \arrow[dr, "z_{p_m+q_2}"'] & q_{2(m+1)} \arrow[d, "z_{q_2}"'] \\
\vdots \arrow[d, "z_{q_{n-1}}"'] \arrow[dr, "z_{p_1+q_{n-1}}"'] & \vdots \arrow[d, "z_{q_{n-1}}"'] \arrow[dr, "z_{p_2+q_{n-1}}"'] & \vdots \arrow[dr, "z_{p_m+q_{n-1}}"'] & \vdots \arrow[d, "z_{q_{n-1}}"'] \\
q_{(n-1)(m+1)+1} \arrow[r, "z_{p_1}"'] \arrow[d, "z_{q_n}"'] \arrow[dr, "z_{p_1+q_n}"'] & q_{(n-1)(m+1)+2} \arrow[r, "z_{p_2}"'] \arrow[d, "z_{q_n}"'] \arrow[dr, "z_{p_2+q_n}"'] & \cdots \arrow[r, "z_{p_m}"'] \arrow[dr, "z_{p_m+q_n}"'] & q_{n(m+1)} \arrow[d, "z_{q_n}"'] \\
q_{n(m+1)+1} \arrow[r, "z_{p_1}"'] & q_{n(m+1)+2} \arrow[r, "z_{p_2}"'] & \cdots \arrow[r, "z_{p_m}"'] & q_{(n+1)(m+1)}
\end{tikzcd}
$$


(q_{nm})
$(q_{(n-1)m+1})$
$(q_{(n-1)m+2})$
(q_{nm})
(q_m)
q_1
z_{p_1}
q_2
z_{p_2}
$\cdots$
$z_{p_{m-1}}$
q_m
z_{p_m}
(q_1)
z_{q_1}
$z_{p_1+q_1}$
z_{q_1}
$z_{p_2+q_1}$
$z_{p_{m-1}+q_1}$
z_{q_1}
$z_{p_m+q_1}$
(q_{2m})
q_{m+1}
z_{p_1}
q_{m+2}
z_{p_2}
$\cdots$
$z_{p_{m-1}}$
q_{2m}
z_{p_m}
(q_{m+1})
z_{q_2}
$z_{p_1+q_2}$
z_{q_2}
$z_{p_2+q_2}$
$z_{p_{m-1}+q_2}$
z_{q_2}
$z_{p_m+q_2}$
$\vdots$
$z_{q_{n-1}}$
$z_{p_1+q_{n-1}}$
$z_{q_{n-1}}$
$z_{p_2+q_{n-1}}$
$z_{p_{m-1}+q_{n-1}}$
$z_{q_{n-1}}$
$z_{p_m+q_{n-1}}$
(q_{mn})
$q_{(n-1)m+1}$
z_{p_1}
$q_{(n-1)m+2}$
z_{p_2}
$\cdots$
$z_{p_{m-1}}$
q_{nm}
z_{p_m}
$(q_{(n-1)m+1})$
z_{q_n}
$z_{p_1+q_n}$
z_{q_n}
$z_{p_2+q_n}$
$z_{p_{m-1}+q_n}$
z_{q_n}
$z_{p_m+q_n}$
(q_1)
(q_2)
$\cdots$
(q_m)
(q_1)

隣接行列を使ったWaldschmidt公式の証明

$$A = \begin{bmatrix} 0 & 0 & x & x \\ 0 & 0 & y & -y \\ y & x & 0 & 0 \\ -y & x & 0 & 0 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} 0 & 2x^2 & 0 & 0 \\ -2y^2 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -4x^2y^2 & 0 & 0 & 0 \\ 0 & -4y^2x^2 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

$$a_{11}^{(n)} = 0 \ (n \neq 4k), \qquad a_{11}^{(4k)} = (-4x^2y^2)^k.$$

また，有限複シャッフル関係式（命題3)，定理4を使うと，

$$Z((xy)^\star ш (-xy)^\star) = Z((xy)^\star * (-xy)^\star) = Z(-(x^3y)^\star)$$

重さ4nのところを拾うと

$$\sum_{p=0}^{2n} (-1)^{2n-p}\zeta(\{2\}_p)\zeta(\{2\}_{2n-p}) = (-1)^n\zeta(\{4\}_n)$$

定理 6 $\mathbf{(}(-xy)^\star ш (xy)^\star x^2 (xy)^\star\mathbf{)}$

$$(i)\qquad \pi^{4n+2}\sum_{p=0}^{n-1}(-1)^{p+1}\frac{4^{2n-p+1}B_{4n-4p+2}}{(4p+2)!(4n-4p+2)!}$$
$$= -2\sum_{p=0}^{n-1}(-4)^p\zeta(\{3,1\}_p,3,3,\{2\}_{2(n-p-1)})$$
$$-3\sum_{p=0}^{n-1}(-4)^p\zeta(\{3,1\}_p,4,\{2\}_{2(n-p)-1})$$
$$+2(-4)^n\sum_{p=0}^{n-1}\zeta(\{3,1\}_p,5,1,\{3,1\}_{n-p-1})$$

重み$= 4n+2$.

$$(ii) \quad \frac{(-1)^n 2\cdot 4^{n+1}(n+1)\pi^{4n+4}}{(4n+6)!} \quad (\text{重み} = 4n+4)$$

$$= 2\sum_{p=0}^{n-1}(-4)^p\zeta(\{3,1\}_p,3,3,\{2\}_{2(n-p)-1})$$

$$+ 3\sum_{p=0}^{n}(-4)^p\zeta(\{3,1\}_p,4,\{2\}_{2(n-p)})$$

$$- 2(-4)^n\sum_{p=0}^{n-1}\zeta(\{3,1\}_p,3,4,1,\{3,1\}_{n-p-1})$$

$$- 2(-4)^n\sum_{p=0}^{n}\zeta(\{3,1\}_p,4,\{3,1\}_{n-p})$$

定理 7 $\left((x^2y)^\star ш (-x^2y)^\star\right)$

$$\sum_{\varepsilon_i,\,\varepsilon'_j=0,1} \frac{12^n}{2^{\varepsilon_1+\cdots+\varepsilon_n+\varepsilon'_1+\cdots+\varepsilon'_{n-1}}}\zeta(\{5-\varepsilon'_{i-1}-\varepsilon_i, 1+\varepsilon_i+\varepsilon'_i\}_{i=1}^n)$$

$$=\zeta(\{6\}_n)=\frac{6(2\pi)^{6n}}{(6n+3)!} \quad (\varepsilon'_0=\varepsilon'_n=0)$$

定理 8 $\left((xy)^\star ш (\omega xy)^\star ш (\omega^2 xy)^\star\right)$

$$\sum_{\varepsilon_i,\ \varepsilon'_j=0,1} \frac{36^n}{3^{\varepsilon_1+\cdots+\varepsilon_n+\varepsilon'_1+\cdots+\varepsilon'_{n-1}}}\zeta(\{4-\varepsilon_i-\varepsilon'_{i-1}, 1+\varepsilon_i, 1+\varepsilon'_i\}_{i=1}^n)$$

$$=\zeta(\{6\}_n) \qquad (\varepsilon'_0=\varepsilon'_n=0)$$

$$\zeta(\{8\}_n) = \frac{2^{6n+2}\pi^{8n}\{(3+2\sqrt{2})^{2n+1}+(3-2\sqrt{2})^{2n+1}\}}{(8n+4)!}\ ,$$

$$\zeta(\{10\}_n) = \frac{5\cdot 2^{8n}\pi^{10n}\{2^{2n+1}+(11+5\sqrt{5})^{2n+1}+(11-5\sqrt{5})^{2n+1}\}}{(10n+5)!}\ ,$$

$$\zeta(\{12\}_n) = \frac{3\cdot 2^{12n+2}\pi^{12n}\{2^{6n+3}+(26+15\sqrt{3})^{2n+1}+(26-15\sqrt{3})^{2n+1}\}}{(12n+6)!}\ .$$

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