A New Learning Algorithm for the Hierarchical Structure Learning Automata Operating in the Nonstationary S-Model Random Environment

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Abstract—An extended algorithm of the relative reward strength algorithm is proposed. It is shown that the proposed algorithm ensures the convergence with probability 1 to the optimal path under the certain type of nonstationary environment. Several computer simulation results confirm the effectiveness of the proposed algorithm.

Index Terms—Hierarchical structure learning automata (HSLA), nonstationary environment, relative reward strength algorithm.

I. INTRODUCTION

THE CONCEPT of learning automata (LAs) operating in an unknown environment was initially introduced by Tsetlin [1]. He considered the learning behaviors of finite deterministic automata in a stationary random environment, and succeeded in proving that they are asymptotically optimal under some conditions.

The study of learning behaviors of LAs originated with Varshavskii and Vorontsova [2] and has since been done extensively by many researchers [3]–[26]. LAs have made a significant impact upon many areas of engineering problems; and have so far been successfully applied to many areas. They are expected to provide one of the most powerful tools for constructing an intelligent system.

Although the study of LAs has matured, there are still several problems to be settled. Two of the most important are the low speed of learning, and the insufficient tracking ability to the changing environment (nonstationary environment). In order to overcome the first problem, the concept of the hierarchical structure automata was originally proposed by Thathachar and Ramakrishnan [8] and Ramakrishnan [9]. Since then, the learning behaviors of the hierarchical structure automata have been extensively studied by many researchers [5], [6], [10]–[12]. To overcome the second problem, Thathachar and Sastry [13] proposed a learning algorithm which uses the average of all the reward responses from the environment. They also extended the use of their algorithm to the hierarchical LA model [14],

[15]. Following their research, Oommen and Lanctot [16] and Papadimitriou [17] introduced the two concepts of discretized pursuit algorithm and stochastic estimator learning algorithm, respectively. Their continuous work has yielded fruitful results [18], [19].

In 1989, Simha and Kurose [20] derived a very interesting algorithm, the approach of which is considerably different from previous algorithms. They proposed the relative reward strength algorithm, which utilizes the most recent reward response from the environment in an intelligent way. They proved that the proposed algorithm converges to the optimal action with probability 1. They also gave several computer simulation results which confirmed the effectiveness of their algorithm, and touched upon the possibility of using it efficiently in a certain type of nonstationary environment.

However, despite the effectiveness of their algorithm, additional studies concerning the learning behaviors of this type of algorithm have not followed. We cannot explain such an omission.

One of the most reasonable ways to use this type of algorithm in an environment with high dimensionality is to utilize the hierarchical system of the LAs. This means that one should extend the original relative reward strength algorithm [20] to be utilized in the hierarchical system of the LAs, and then carefully investigate its learning performance. However, unfortunately, this has not yet been attempted.

In this paper, we shall try to extend the algorithm of Simha and Kurose to be used in the hierarchical structure learning automata (HSLA) model. We shall offer evidence that our extended algorithm converges to the optimal action under the certain type of the nonstationary S-model environment. We shall also give several computer simulation results which confirm the effectiveness of our extended algorithm.

II. BASIC MODEL OF A LEARNING AUTOMATON OPERATING IN AN UNKNOWN ENVIRONMENT

In the following, we will discuss the learning behaviors of the HSLA operating in the nonstationary S-model environment. To place our study in the appropriate context, we start with a brief explanation of the basic model of the learning mechanism of the single automaton in the stationary random environment.

The learning behaviors of a variable-structure LA operating in an unknown random environment have been discussed extensively under the basic model shown in Fig. 1. Let us

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briefly explain the learning mechanism of the LA "A" under the unknown random environment (teacher environment) $R(C_1, C_2, \dots, C_r)$.

The LA "A" is defined by the sextuple $\{S, W, Y, g, P(t), T\}$. S denotes the set of two inputs (0,1), where "1" indicates the reward response from $R(C_1, C_2, \dots, C_r)$, and "0" indicates the penalty response. (If the set S consists of only the two elements 0 and 1, the environment is said to be a P-model. When the input into A assumes a finite number of values in the closed interval [0,1], it is said to be a Q-model. An S-model is one in which the input into A takes an arbitrary number in the closed line segment [0,1]. In this paper, we will deal with the S-model environment.) W denotes the set of r internal states $(w_1, w_2, \dots, w_r)$; Y denotes the set of r outputs $(y_1, y_2, \dots, y_r)$; g denotes the output function $y(t) = g[w(t)]$, that is, one-to-one deterministic mapping; $P(t)$ denotes the probability vector $[p_1(t), p_2(t), \dots, p_r(t)]'$ at time t ; and its i th component $p_i(t)$ indicates the probability with which the i th state w_i is chosen at time t ($i = 1, 2, \dots, r$)

$$p_1(0) = p_2(0) = \dots = p_r(0) = \frac{1}{r}, \quad \sum_{i=1}^r p_i(t) = 1$$

and T denotes the reinforcement scheme which generates $P(t+1)$ from $P(t)$.

Suppose that the state w_i is chosen at time t . Then, the LA "A" performs action y_i on the random environment $R(C_1, C_2, \dots, C_r)$. In response to the action y_i , the environment emits output $s(t) = 1$ (reward) with probability $1 - C_i$; and output $s(t) = 0$ (penalty) with probability C_i ($i = 1, 2, \dots, r$). If all of the C_i ($i = 1, 2, \dots, r$) are constant, the random environment $R(C_1, C_2, \dots, C_r)$ is said to be a stationary random environment. (The term "single teacher environment" is also used.) On the other hand, if C_i ($i = 1, 2, \dots, r$) are not constant, it is said to be a nonstationary random environment. Depending upon the action of the LA "A" and the environmental response to it, the reinforcement scheme T changes the probability vector $P(t)$ to $P(t+1)$.

The values of C_i ($i = 1, 2, \dots, r$) are not known in advance. Therefore, it is necessary to reduce the average penalty

$$M(t) = \sum_{i=1}^r p_i(t) C_i$$

by selecting an appropriate reinforcement scheme. To judge the effectiveness of an LA operating in a stationary random environment $R(C_1, C_2, \dots, C_r)$, various performance measures such as optimality, ε -optimality, absolute expediency, etc., have been set up. (Details are omitted due to space. The reader is referred to [3]–[7].)

In this section, we have briefly introduced the learning mechanism of the single automaton under the stationary random environment $R(C_1, C_2, \dots, C_r)$. However, when applying LAs to various actual problems, one often encounters hard situations where nonstationary random environment [5]–[7], [22], [23], multiteacher environment [5], [7], [24], HSLA model [5]–[10],

Fig. 1. Basic model of an LA operating in an unknown random environment.

[12], [14], [15], [18], [25], etc., must be considered. In the following section, we will explain the learning mechanism of the HSLA operating in the nonstationary S-model environment.

III. HIERARCHICAL STRUCTURE LEARNING AUTOMATA (HSLA) OPERATING IN THE NONSTATIONARY S-MODEL ENVIRONMENT

After the pioneering work of Tsetlin [1], learning behaviors of the single automaton have been extensively studied under the basic model shown in Fig. 1. However, one of the most serious bottlenecks concerning the learning model of the single automaton is that its learning performance declines considerably when the space of decisions has high dimensionality.

In order to overcome this problem, Thathachar and Ramakrishnan [8] proposed the concept of the HSLA. Since then, many active researchers have been involved in the study of the learning behaviors of the HSLA.

Next, we consider the learning behaviors of the HSLA operating in the nonstationary S-model environment.

A. Hierarchical Structure Learning Automata (HSLA)

The system of the HSLA considered in this paper is briefly described as follows. (For easier understanding, the system of the HSLA with the hierarchy of three levels is shown in Fig. 2.) The hierarchy consists of the tree-structured LAs each with r actions. Z denotes the top level (first level) automaton. $Z_1, Z_2, \dots, Z_r$ denote the automata at the second level. $Z_{11}, Z_{12}, \dots, Z_{rr-1}, Z_{rr}$ denote the automata at the third level. At the general s th ($s = 1, 2, \dots, N$) level, there are r^{s-1} automata $Z_{i_1 i_2 \dots i_{s-1}}$.

Each LA $Z_{i_1 i_2 \dots i_{s-1}}$ ($s = 1, 2, \dots, N$) is defined by $\{\alpha_{i_1 i_2 \dots i_{s-1}}, \beta(t), p_{i_1 i_2 \dots i_{s-1}}(t), T, v_{i_1 i_2 \dots i_{s-1}}(t)\}$. Here, $\alpha_{i_1 i_2 \dots i_{s-1}} = \{\alpha_{i_1 i_2 \dots i_{s-1}1}, \alpha_{i_1 i_2 \dots i_{s-1}2}, \dots, \alpha_{i_1 i_2 \dots i_{s-1}r}\}$ denotes the set of r actions, $\beta(t)$ denotes the reward response from the environment, $p_{i_1 i_2 \dots i_{s-1}}(t) = (p_{i_1 i_2 \dots i_{s-1}1}(t), p_{i_1 i_2 \dots i_{s-1}2}(t), \dots, p_{i_1 i_2 \dots i_{s-1}r}(t))'$ denotes the action probability distribution, T denotes the reinforcement scheme, $v_{i_1 i_2 \dots i_{s-1}}(t) = (v_{i_1 i_2 \dots i_{s-1}1}(t), v_{i_1 i_2 \dots i_{s-1}2}(t), \dots, v_{i_1 i_2 \dots i_{s-1}r}(t))'$ denotes the recent reward vector.¹

Let us now examine how the hierarchical system behaves. Initially, all the action probabilities for each automaton at each level are set equal. The first level automaton Z chooses an action at time t from the action probability distribution $p(t) = (p_1(t), p_2(t), \dots, p_r(t))'$. Suppose that α_{j_1} is the output from

¹The recent reward vector will be defined in the next section.

Fig. 2. Hierarchical structure learning automata (HSLA).

Z . Then, at the second level, the automaton Z_{j_1} is actuated. The Z_{j_1} also chooses an action from its action probability distribution. Assume that $\alpha_{j_1 j_2}$ is the output from Z_{j_1} . Then, at the third level, the automaton $Z_{j_1 j_2}$ is actuated. This cycle of operation is repeated from the top (first level) to the bottom (N th level). The action at the lowest level interacts with the environment.

The sequence of actions $\{\alpha_{j_1}, \alpha_{j_1 j_2}, \dots, \alpha_{j_1 j_2 \dots j_N}\}$ having been chosen by N automata is called the path. Let $\phi_{j_1 j_2 \dots j_N}$ denote the path. Corresponding to the path, the HSLA model receives reward strength $\beta(t)^2$ as an environmental response. The HSLA model utilizes this $\beta(t)$ in order to update the current recent reward vector. The action probability vector of each LA relating to the path is updated by using the information concerning the recent reward vector. After all of the above procedures have been completed, time t is set to be $t+1$. By repeating the above cycle of operation, learning by the hierarchical structure automata proceeds. Let $\pi_{j_1 j_2 \dots j_N}(t)$ denote the probability that the path $\phi_{j_1 j_2 \dots j_N}$ is chosen at time t . From this definition

$$\pi_{j_1 j_2 \dots j_N}(t) = p_{j_1}(t) p_{j_1 j_2}(t) \dots p_{j_1 j_2 \dots j_N}(t). \quad (1)$$

B. The Nonstationary S-Model Environment

Next, we consider the learning behaviors of the hierarchical structure automata model under the S-model nonstationary environment with the following property.

The optimal path $\phi_{j_1^* j_2^* \dots j_N^*}$ satisfying the following relation exists uniquely³:

$$\min_t E\{\beta_{j_1^* j_2^* \dots j_N^*}(t)\} > \max_t E\{\beta_{i_1 i_2 \dots i_N}(t)\} \quad (2)$$

where $\beta_{i_1 i_2 \dots i_N}$ denotes the environmental response corresponding to the path $\phi_{i_1 i_2 \dots i_N}$ and $\max\{|j_1^* - i_1|, |j_2^* - i_2|, \dots, |j_N^* - i_N|\} > 0$.

The objective for the hierarchical structure of LA is to find the optimal path $\phi_{j_1^* j_2^* \dots j_N^*}$ with the probability as high as possible.

² $\beta(t)$ can be an arbitrary number in the closed line segment $[0,1]$. The large value of $\beta(t)$ indicates that the high reward is given to the sequence of actions being chosen.

³ $E\{\bullet\}$ denotes the mathematical expectation of $\bullet$.

Remark 1: The inequality (2) indicates that the minimum value of the average reward from the environment corresponding to the optimal path is the larger than the maximum value of the average reward corresponding to the other path. This means that we will discuss the learning performance of the proposed algorithm under the nonstationary S-model environment, the constraint of which is rather strict.

IV. A NEW LEARNING ALGORITHM OF THE HIERARCHICAL STRUCTURE AUTOMATA MODEL

In 1989, Simha and Kurose [20] proposed a new class of update algorithms which realize a sophisticated use of recent environmental responses. They showed that their algorithms are suitable for tracking nonstationary behaviors of environments. In this section, we shall extend their algorithms in order to use them effectively in the HSLA system.

First, let us define “recent reward to the path” and “recent reward vector.”

Definition 1: Let us assume that the path $\phi_{i_1 i_2 \dots i_N}$ has been chosen at time t and, corresponding to $\phi_{i_1 i_2 \dots i_N}$, reward $\beta_{i_1 i_2 \dots i_N}$ has been given from the environment. Then, “recent reward to the path $\phi_{i_1 i_2 \dots i_N}$ ” is defined as follows:

$$u_{i_1 i_2 \dots i_N}(t) = \beta_{i_1 i_2 \dots i_N}. \quad (3)$$

On the other hand, the other recent reward to the path $\phi_{j_1 j_2 \dots j_N}$ ($\exists k; i_k \neq j_k$) is defined as follows:

$$u_{j_1 j_2 \dots j_N}(t) = \beta_{j_1 j_2 \dots j_N}(\tau_{j_1 j_2 \dots j_N}) \quad (4)$$

where $\tau_{j_1 j_2 \dots j_N}$ is the most recent time when the path $\phi_{j_1 j_2 \dots j_N}$ has been chosen, and $\beta_{j_1 j_2 \dots j_N}(\tau_{j_1 j_2 \dots j_N})$ is the reward from the environment at $\tau_{j_1 j_2 \dots j_N}$.

Definition 2: Let $\mathbf{v}_{i_1 i_2 \dots i_{s-1}}(t) = (v_{i_1 i_2 \dots i_{s-1} 1}(t), v_{i_1 i_2 \dots i_{s-1} 2}(t), \dots, v_{i_1 i_2 \dots i_{s-1} r}(t))'$ be the recent reward vector corresponding to the s th level LA $Z_{i_1 i_2 \dots i_{s-1}}$ ($s = 1, 2, \dots, N$).

Here, each of the components of $\mathbf{v}_{i_1 i_2 \dots i_{s-1}}(t)$ is constructed as follows:

at the N th level

$$v_{i_1 i_2 \dots i_N}(t) = u_{i_1 i_2 \dots i_N}(t). \quad (5)$$

at the s th level ($1 \leq s \leq N-1$)

$$v_{i_1 i_2 \dots i_s}(t) = \max\{v_{i_1 i_2 \dots i_{s-1} 1}(t), v_{i_1 i_2 \dots i_{s-1} 2}(t), \dots, v_{i_1 i_2 \dots i_{s-1} r}(t)\}. \quad (6)$$

As in [20], we also assume that the following condition holds for all $i_1, i_2, \dots, i_s$ ($i_q = 1, 2, \dots, r; q = 1, 2, \dots, s$ ($s = 1, 2, \dots, N$)):

$$q_{\min} \leq p_{i_1 i_2 \dots i_s}(t) \leq q_{\max} \quad (7)$$

where $q_{\min}$ and $q_{\max}$ satisfy the inequalities $0 < q_{\min} < q_{\max} < 1$ and $q_{\max} = 1 - (r-1)q_{\min}$, respectively.

Next, we shall propose a new learning algorithm which is an extended form of the relative reward strength algorithm of Simha and Kurose [20]

A. Learning Algorithm

Assume that the path $\phi(t) = \phi_{j_1 j_2 \dots j_N}$ has been chosen at time t and (corresponding to the output from the hierarchical structure automata system) the environmental response $u_{j_1 j_2 \dots j_N}$ has been given. Then, the action probabilities $p_{j_1 j_2 \dots j_{s-1} i_s}(t)$ ($i_s = 1, 2, \dots, r$) of each automaton $Z_{j_1 j_2 \dots j_{s-1}}$ ($s = 1, 2, \dots, N$) connected to the path being chosen are updated by the following equation:

$$p_{j_1 j_2 \dots j_{s-1} i_s}(t+1) = p_{j_1 j_2 \dots j_{s-1} i_s}(t) + \lambda_{j_1 j_2 \dots j_{s-1}}(t) \Delta p_{j_1 j_2 \dots j_{s-1} i_s}(t) \quad (8)$$

where $\Delta p_{j_1 j_2 \dots j_{s-1} i_s}(t)$ is calculated by

$$\Delta p_{j_1 j_2 \dots j_{s-1} i_s}(t) = \begin{cases} \frac{v_{j_1 j_2 \dots j_{s-1} i_s}(t)}{|A_s(t)|} \sum_{l_s \in A_s(t)} v_{j_1 j_2 \dots j_{s-1} l_s}(t) & \forall i_s \in A_s(t) \\ 0, & i_s \notin A_s(t). \end{cases} \quad (9)$$

Here, the set $A_s(t)$ is constructed as follows.

- 1) Line up $v_{j_1 j_2 \dots j_{s-1} i_s}(t)$ in descending order.
- 2) Set $D_{j_1 j_2 \dots j_{s-1}} = \{k_s | v_{j_1 j_2 \dots j_{s-1} k_s}(t) = \max_{i_s} \{v_{j_1 j_2 \dots j_{s-1} i_s}(t)\}\}$.
- 3) Repeat the following procedure for i_s ($i_s \notin D_{j_1 j_2 \dots j_{s-1}}$) in descending order of $v_{j_1 j_2 \dots j_{s-1} i_s}(t)$: If the inequality $p_{j_1 j_2 \dots j_{s-1} i_s}(t+1) > q_{\min}$ can be satisfied as a result of calculation by (8) and (9), then set $D_{j_1 j_2 \dots j_{s-1}} = D_{j_1 j_2 \dots j_{s-1}} \cup \{i_s\}$.
- 4) Set $A_s(t) = D_{j_1 j_2 \dots j_{s-1}}$.

Remark 2: $\lambda_{j_1 j_2 \dots j_{s-1}}(t)$ is the step size parameter at time t .

Remark 3: In the proposed algorithm, the change in the action probability of each actuated automaton in the hierarchy is proportional to the difference between the corresponding component of the recent reward vector and the average recent reward over all actions in the actuated automaton.

Remark 4: The action probabilities of each automaton which is not on the selected path are not changed.

Remark 5: As we have already mentioned, the proposed algorithm has been obtained by considering the extension of the relative reward strength algorithm proposed by Simha and Kurose [20]. This means that the proposed algorithm coincides with the original one when $N = 1$. Several computer simulation results [20] showed that the relative reward strength algorithm (used in the single automaton) outperforms the SL_{R-I} scheme under the nonstationary random environment with high noise.

V. CONVERGENCE THEOREM

In this section, we shall derive a convergence theorem concerning the learning performance of our proposed algorithm. First, we can obtain the following lemma by paying attention to the components of the reward vector $\mathbf{v}_{j_1^* j_2^* \dots j_{s-1}^*}(t) = (v_{j_1^* j_2^* \dots j_{s-1}^* 1}(t), v_{j_1^* j_2^* \dots j_{s-1}^* 2}(t), \dots, v_{j_1^* j_2^* \dots j_{s-1}^* r}(t))'$ ($s = 1, 2, \dots, N$) corresponding to the optimal path $\phi_{j_1^* j_2^* \dots j_N^*}$.

Lemma 1: Suppose that each component of the current reward vector is given by (5) and (6). Then, the following inequality concerning the LA $Z_{j_1^* j_2^* \dots j_{s-1}^*}$ ($s = 1, 2, \dots, N$) holds:

$$E\{v_{j_1^* j_2^* \dots j_s^*}(t)\} > E\{v_{j_1^* j_2^* \dots j_{s-1}^* i_s}(t)\} \quad (10)$$

where $i_s = 1, 2, \dots, r$, $i_s \neq j_s^*$.

Proof: Let us carefully examine each of the LAs relating to the optimal path $\phi_{j_1^* j_2^* \dots j_N^*}$. From (4) and (5), we can derive the following equality concerning the LA $Z_{j_1^* j_2^* \dots j_{N-1}^*}$ at the lowest level N :

$$\begin{aligned} E\{v_{j_1^* j_2^* \dots j_{N-1}^* i_N}(t)\} &= E\{u_{j_1^* j_2^* \dots j_{N-1}^* i_N}(t)\} \\ &= E\{\beta_{j_1^* j_2^* \dots j_{N-1}^* i_N}(\tau_{j_1^* j_2^* \dots j_{N-1}^* i_N})\} \\ &\quad (i_N = 1, 2, \dots, r). \end{aligned} \quad (11)$$

Therefore, from (2) of the optimal path, we can arrive at the following relation:

$$\begin{aligned} E\{v_{j_1^* j_2^* \dots j_{N-1}^* j_N^*}(t)\} &= E\{\beta_{j_1^* j_2^* \dots j_{N-1}^* j_N^*}(\tau_{j_1^* j_2^* \dots j_{N-1}^* j_N^*})\} \\ &> E\{\beta_{j_1^* j_2^* \dots j_{N-1}^* i_N}(\tau_{j_1^* j_2^* \dots j_{N-1}^* i_N})\} \\ &= E\{v_{j_1^* j_2^* \dots j_{N-1}^* i_N}(t)\} \\ &\quad (i_N = 1, 2, \dots, r; i_N \neq j_N^*). \end{aligned} \quad (12)$$

Now, let us consider the LA $Z_{j_1^* j_2^* \dots j_{N-2}^*}$ at the level $(N-1)$. From (6) and (12), we can get

$$\begin{aligned} E\{v_{j_1^* j_2^* \dots j_{N-2}^* j_{N-1}^*}(t)\} &= \max_{l_N} \{E\{v_{j_1^* j_2^* \dots j_{N-1}^* l_N}(t)\}\} \\ &= E\{v_{j_1^* j_2^* \dots j_{N-1}^* j_N^*}(t)\} \\ &= E\{\beta_{j_1^* j_2^* \dots j_{N-1}^* j_N^*}(\tau_{j_1^* j_2^* \dots j_{N-1}^* j_N^*})\} \\ &\quad (13) \\ E\{v_{j_1^* j_2^* \dots j_{N-2}^* i_{N-1}}(t)\} &= \max_{l_N} \{E\{v_{j_1^* j_2^* \dots j_{N-2}^* i_{N-1} l_N}(t)\}\} \\ &= E\{v_{j_1^* j_2^* \dots j_{N-2}^* i_{N-1} k_N^*}(t)\} \\ &= E\{\beta_{j_1^* j_2^* \dots j_{N-2}^* i_{N-1} k_N^*}(\tau_{j_1^* j_2^* \dots j_{N-2}^* i_{N-1} k_N^*})\} \\ &\quad (i_{N-1} = 1, 2, \dots, r; i_{N-1} \neq j_{N-1}^*). \end{aligned} \quad (14)$$

If $i_{N-1} \neq j_{N-1}^*$, we can show the following relation from (13) and (14):

$$\begin{aligned} E\{v_{j_1^* j_2^* \dots j_{N-2}^* j_{N-1}^*}(t)\} &= E\{\beta_{j_1^* j_2^* \dots j_{N-1}^* j_N^*}(\tau_{j_1^* j_2^* \dots j_{N-1}^* j_N^*})\} \\ &> E\{\beta_{j_1^* j_2^* \dots j_{N-2}^* i_{N-1} k_N^*}(\tau_{j_1^* j_2^* \dots j_{N-2}^* i_{N-1} k_N^*})\} \\ &= E\{v_{j_1^* j_2^* \dots j_{N-2}^* i_{N-1}}(t)\} \\ &\quad (i_{N-1} = 1, 2, \dots, r; i_{N-1} \neq j_{N-1}^*). \end{aligned} \quad (15)$$

Furthermore, let us consider the LA $Z_{j_1^* j_2^* \dots j_{N-3}^*}$ at the level $(N-2)$. From (6), we can get

$$\begin{aligned}
 & E\{v_{j_1^* j_2^* \dots j_{N-2}^*}(t)\} \\
 &= \max_{l_{N-1}, l_N} \{E\{v_{j_1^* j_2^* \dots j_{N-2}^* l_{N-1} l_N}(t)\}\} \\
 &= E\{\beta_{j_1^* j_2^* \dots j_{N-2}^* j_{N-1}^* j_N^*}(\tau_{j_1^* j_2^* \dots j_{N-2}^* j_{N-1}^* j_N^*})\} \\
 & E\{v_{j_1^* j_2^* \dots j_{N-3}^* i_{N-2}}(t)\} \\
 &= \max_{l_{N-1}, l_N} \{E\{v_{j_1^* j_2^* \dots j_{N-3}^* i_{N-2} l_{N-1} l_N}(t)\}\} \\
 &= E\{\beta_{j_1^* j_2^* \dots j_{N-3}^* i_{N-2} k_{N-1}^* k_N^*}(\tau_{j_1^* j_2^* \dots j_{N-3}^* i_{N-2} k_{N-1}^* k_N^*})\} \\
 & \quad (i_{N-2} = 1, 2, \dots, r; i_{N-2} \neq j_{N-2}^*). \quad (17)
 \end{aligned}$$

Therefore, from (2), (16), and (17), we can get the following relation:

$$\begin{aligned}
 & E\{v_{j_1^* j_2^* \dots j_{N-2}^*}(t)\} \\
 &= E\{\beta_{j_1^* j_2^* \dots j_{N-2}^* j_{N-1}^* j_N^*}(\tau_{j_1^* j_2^* \dots j_{N-2}^* j_{N-1}^* j_N^*})\} \\
 &> E\{\beta_{j_1^* j_2^* \dots j_{N-3}^* i_{N-2} k_{N-1}^* k_N^*}(\tau_{j_1^* j_2^* \dots j_{N-3}^* i_{N-2} k_{N-1}^* k_N^*})\} \\
 &= E\{v_{j_1^* j_2^* \dots j_{N-3}^* i_{N-2}}(t)\} \\
 & \quad (i_{N-2} = 1, 2, \dots, r; i_{N-2} \neq j_{N-2}^*). \quad (18)
 \end{aligned}$$

By repeating the same procedures, we can easily prove that the following inequality holds at an arbitrary level s ($s = 1, 2, \dots, N$)

$$E\{v_{j_1^* j_2^* \dots j_s^*}(t)\} > E\{v_{j_1^* j_2^* \dots j_{s-1}^* i_s}(t)\} \quad (i_s = 1, 2, \dots, r; i_s \neq j_s^*).$$

Q.E.D.

By taking advantage of Lemma 1, we can obtain the following theorem concerning the convergence to the optimal path $\phi_{j_1^* j_2^* \dots j_N^*}$.

Theorem 1: Assume that the condition (7) and the conditions given in the lemma hold. Furthermore, let $\lambda_{j_1 j_2 \dots j_{s-1}}(t)$ be a sequence of real numbers such that

$$\begin{aligned}
 & \lambda_{j_1 j_2 \dots j_{s-1}}(t) > 0, \quad \sum_{t=1}^{\infty} \lambda_{j_1 j_2 \dots j_{s-1}}(t) = \infty \\
 & \sum_{t=1}^{\infty} \lambda_{j_1 j_2 \dots j_{s-1}}^2(t) < \infty. \quad (19)
 \end{aligned}$$

Then, the path probability $\pi_{j_1^* j_2^* \dots j_N^*}(t)$ that the hierarchical structure automata system chooses the optimal path $\phi_{j_1^* j_2^* \dots j_N^*}$ at time t converges almost surely to $(q_{\max})^N$.

Proof: From the lemma, it can be shown that $E\{v_{j_1^* j_2^* \dots j_{s-1}^* j_s^*}(t)\}$ corresponding to the optimal action $\alpha_{j_1^* j_2^* \dots j_{s-1}^* j_s^*}$ (related to the optimal path) is always greater than $E\{v_{j_1^* j_2^* \dots j_{s-1}^* i_s}(t)\}$ corresponding to the other action

$\alpha_{j_1^* j_2^* \dots j_{s-1}^* i_s}(i_s \neq j_s^*)$. This means that the assumption of Theorem 1 [20] concerning the average reward holds. Therefore, it can be shown that by using the same procedure as in Theorem 1 [20]

$$\lim_{t \rightarrow \infty} p_{j_1^* j_2^* \dots j_s^*}(t) = q_{\max} \text{ with probability 1} \quad (20)$$

for all of the LAs $Z_{j_1^* j_2^* \dots j_{s-1}^*}$ ($s = 1, 2, \dots, N$) related to the optimal path (given in the Appendix). The path probability is thus given by (1). Hence, from (20), we can obtain

$$\begin{aligned}
 & \lim_{t \rightarrow \infty} \pi_{j_1^* j_2^* \dots j_N^*}(t) \\
 &= \lim_{t \rightarrow \infty} \{p_{j_1^*}(t) p_{j_1^* j_2^*}(t) \dots p_{j_1^* j_2^* \dots j_N^*}(t)\} \\
 &= \lim_{t \rightarrow \infty} p_{j_1^*}(t) \lim_{t \rightarrow \infty} p_{j_1^* j_2^*}(t) \dots \lim_{t \rightarrow \infty} p_{j_1^* j_2^* \dots j_N^*}(t) \\
 &= q_{\max} q_{\max} \dots q_{\max} = (q_{\max})^N. \quad (21)
 \end{aligned}$$

Q.E.D.

VI. COMPUTER SIMULATION RESULTS

In order to investigate whether the proposed algorithm can be successfully utilized in the nonstationary S-model environments, we carried out many computer simulations.

We shall show one of the computer simulation results concerning the learning behaviors of the HSLA.

Before going into details concerning the computer simulation, we shall briefly explain the hierarchical automata system, nonstationary environments, etc.

A. Hierarchical Structure Automata Model

The hierarchical structure automata model is characterized by the following:

- 1) the number of the levels of the HSLA: 11;
- 2) the number of the actions of each automaton in the hierarchy: 2;
- 3) the total number of paths: 2048;
- 4) the optimal path: $\phi_{11 \ 111 \ 221 \ 121}$.

B. Nonstationary Environment

We have considered the following nonstationary environment:

- 1) The environmental reward $\beta(t)$ at time t corresponding to the output $\phi(t) = \phi_{i_1 i_2 \dots i_N}$ from the hierarchy is characterized by the following equation:

$$\beta(t) = \beta_{i_1 i_2 \dots i_N}(t) + e_{i_1 i_2 \dots i_N} \xi \quad (22)$$

where ξ is the random variable with the uniform probability density function in the closed interval $[-0.5, 0.5]$ and $\beta_{i_1 i_2 \dots i_N}(t) = a_{i_1 i_2 \dots i_N} + b_{i_1 i_2 \dots i_N} \sin(c_{i_1 i_2 \dots i_N} \pi t + d_{i_1 i_2 \dots i_N})$. Here, $a_{i_1 i_2 \dots i_N}$, $b_{i_1 i_2 \dots i_N}$, $c_{i_1 i_2 \dots i_N}$, $d_{i_1 i_2 \dots i_N}$, and $e_{i_1 i_2 \dots i_N}$ are the positive scalars, the values of which have been chosen in such a way that the inequality $0 \leq \beta(t) \leq 1$ holds for all t .

TABLE I
VALUES OF THE COEFFICIENTS $a_{i_1 i_2 \dots i_N}$ AND $b_{i_1 i_2 \dots i_N}$

path	a	b
Optimal Path		
$\phi_{11111221121}$	0.88	0.05
Second optimal path		
$\phi_{11111112122}$	0.70	0.08

TABLE II
SIMULATION RESULTS (OUR ALGORITHM)

Step size parameters	Average Number of Iterations	Percentage of correct convergences(%)
0.01	2234.0	100
0.02	1188.2	100
0.03	851.4	100
0.04	8640.5	100
0.05	11944.1	100

2) In the simulation, we utilized the following particular combinations concerning the parameter values:

- $a_{i_1 i_2 \dots i_N}$, $b_{i_1 i_2 \dots i_N}$: The values of $a_{i_1 i_2 \dots i_N}$ and $b_{i_1 i_2 \dots i_N}$ corresponding, respectively, to the optimal path and the second optimal path are given in Table I. The values of the other $a_{i_1 i_2 \dots i_N}$ have been chosen by using the random variable with the uniform probability density function in the closed interval $[0.1, 0.5]$. On the other hand, the values of the other $b_{i_1 i_2 \dots i_N}$ have been chosen to be equal to the same value 0.1.
- $c_{i_1 i_2 \dots i_N}$: The value of $c_{i_1 i_2 \dots i_N}$ has been chosen by using the random variable with the uniform probability density function in the closed interval $[0.3, 0.8]$.
- $d_{i_1 i_2 \dots i_N}$: 0.5 for all $i_1, i_2, \dots, i_N$.
- $e_{i_1 i_2 \dots i_N}$: 0.03 for all $i_1, i_2, \dots, i_N$.

C. Parameters $q_{\min}$ and $q_{\max}$

As the parameters $q_{\min}$ and $q_{\max}$, we have used the following values:

- 1) $q_{\min} = 0.002$
- 2) $q_{\max} = 0.998$.

We have used the same value of the parameter $\lambda_{i_1 i_2 \dots i_{s-1}} (s = 1, 2, \dots, N)$ from the top level of the hierarchy to the bottom level.

Table II shows the average number of iteration and the probability that the proposed algorithm has succeeded in finding the optimal path. In each computer experiment, we have carried out 30 simulations. In order to compare our proposed algorithm with the familiar learning algorithm by Thathachar and Ramakrishnan [8], we have also carried out computer simulations using their algorithm by keeping the same experimental conditions. Table III shows the computer simulation results. From these results, we may confirm the effectiveness of our proposed algorithm.

TABLE III
SIMULATION RESULTS (T&R ALGORITHM)

Step size parameters	Average Number of Iterations	Percentage of correct convergences(%)
0.00025	127132.4	100
0.00026	131674.1	100
0.00027	126888.2	91
0.00028	126098.2	100
0.00029	113611.0	91
0.0025	11398.4	37
0.0026	10978.6	23
0.0027	12277.0	30
0.0028	11268.2	30
0.0029	10433.4	23

VII. CONCLUSIONS

In this paper, we have extended the relative reward strength algorithm of Simha and Kurose [20] in order that it can be used in the HSLA model. We have indicated that the proposed algorithm ensures convergence to the optimal action with probability 1 under the certain type of nonstationary S-model environment. Future research is needed to investigate the learning behaviors of the hierarchical structure automata under various types of the nonstationary environments.

APPENDIX

Here, we give similar definitions to those in [20].

$$\begin{aligned}
 I_B &= \left[q_{\min}, q_{\max} - \frac{\varepsilon}{2} \right] \\
 I_C &= \left[q_{\max} - \frac{\varepsilon}{2}, q_{\max} \right] \\
 L(p_{j_1^* j_2^* \dots j_s^*}(t)) &= \begin{cases} q_{\max} - \frac{\varepsilon}{2} - p_{j_1^* j_2^* \dots j_s^*}(t) & p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B \\ 0, & p_{j_1^* j_2^* \dots j_s^*}(t) \in I_C. \end{cases} \\
 P_{sB}(t) &= \Pr(p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B) \\
 P_{sC}(t) &= \Pr(p_{j_1^* j_2^* \dots j_s^*}(t) \in I_C) \\
 b_s(t) &= E \{ L^2(p_{j_1^* j_2^* \dots j_s^*}(t)) \} \\
 d_s(t) &= E \left\{ \left(q_{\max} - \frac{\varepsilon}{2} - p_{j_1^* j_2^* \dots j_s^*}(t) \right) \Delta p_{j_1^* j_2^* \dots j_s^*}(t) \right. \\
 &\quad \left. \middle| p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B \right\} \\
 e_s(t) &= E \{ \Delta^2 p_{j_1^* j_2^* \dots j_s^*}(t) | p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B \} \\
 f_s(t) &= E \{ L^2(p_{j_1^* j_2^* \dots j_s^*}(t+1)) | p_{j_1^* j_2^* \dots j_s^*}(t) \in I_C \}. \quad (A.1)
 \end{aligned}$$

Let us pay the attention to the LA $Z_{j_1^* j_2^* \dots j_{s-1}^*} (s = 1, 2, \dots, N)$ connected to the optimal path. Concerning the action probability $p_{j_1^* j_2^* \dots j_s^*}(t)$ related to this automaton, we can prove the following lemma:

Lemma A1: For an arbitrary positive number ε , there exists $\bar{T}(>0)$, and the following inequality (A.2) holds for all $t(>\bar{T})$:

$$E\{\Delta p_{j_1^* j_2^* \dots j_s^*}(t) | p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B\} > 0. \quad (A.2)$$

Proof of Lemma A1: By introducing the set $A_s(t)$, we can get

$$\begin{aligned}
& E\{\Delta p_{j_1^* j_2^* \dots j_s^*}(t) | p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B\} \\
&= E\{\Delta p_{j_1^* j_2^* \dots j_s^*}(t) | p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B \wedge j_s^* \in A_s(t)\} \\
&\quad \times \Pr(j_s^* \in A_s(t)) \\
&\quad + E\{\Delta p_{j_1^* j_2^* \dots j_s^*}(t) | p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B \wedge j_s^* \notin A_s(t)\} \\
&\quad \times \Pr(j_s^* \notin A_s(t)) \\
&= E\{\Delta p_{j_1^* j_2^* \dots j_s^*}(t) | p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B \wedge j_s^* \in A_s(t)\} \\
&\quad \times \Pr(j_s^* \in A_s(t)). \tag{A.3}
\end{aligned}$$

Now let us show that each term of (A.3) is positive. First, we shall consider the first term $E\{\Delta p_{j_1^* j_2^* \dots j_s^*}(t) | p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B \wedge j_s^* \in A_s(t)\}$. From the fact $v_{j_1 j_2 \dots j_s}(t)$ is bounded, and from (9), it can be easily shown that $|\Delta p_{j_1 j_2 \dots j_s}(t)|$ is bounded. This means that there exists $\overline{\Delta p}$ such that $|\Delta p_{j_1 j_2 \dots j_s}(t)| < \overline{\Delta p}$. Furthermore, from (19), $\lambda_{j_1 j_2 \dots j_{s-1}}(t) \rightarrow 0$ ($t \rightarrow \infty$). Therefore, there exists some $\overline{T} > 0$ for an arbitrary positive number ε and the following inequality (A.4) holds for any $t(>\overline{T})$:

$$\begin{aligned}
& \lambda_{j_1 j_2 \dots j_{s-1}}(t) |\Delta p_{j_1 j_2 \dots j_s}(t)| \leq \lambda_{j_1 j_2 \dots j_{s-1}}(t) \overline{\Delta p} \\
& \leq \frac{\varepsilon}{2(r-1)}. \tag{A.4}
\end{aligned}$$

Hence, from (8) and (A.4), we can show

$$\begin{aligned}
& p_{j_1^* j_2^* \dots j_{s-1}^* i_s}(t+1) \\
&= p_{j_1^* j_2^* \dots j_{s-1}^* i_s}(t) + \lambda_{j_1^* j_2^* \dots j_{s-1}^*}(t) \Delta p_{j_1^* j_2^* \dots j_{s-1}^* i_s}(t) \\
&\geq p_{j_1^* j_2^* \dots j_{s-1}^* i_s}(t) - \frac{\varepsilon}{2(r-1)}, \quad t > \overline{T}. \tag{A.5}
\end{aligned}$$

Since $p_{j_1^* j_2^* \dots j_s^*}(t) \leq q_{\max} - (\varepsilon/2)$ ($p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B$), there exists at least one action $\alpha_{j_1^* j_2^* \dots j_{s-1}^* i_s}(i_s \neq j_s^*)$ which satisfies

$$p_{j_1^* j_2^* \dots j_{s-1}^* i_s}(t) \geq q_{\min} + \frac{\varepsilon}{2(r-1)}.$$

From this fact and (A.5), we can easily show that there exists at least one action $\alpha_{j_1^* j_2^* \dots j_{s-1}^* i_s}(i_s \neq j_s^*)$ for any $t(>\overline{T})$ which satisfies

$$\begin{aligned}
& p_{j_1^* j_2^* \dots j_{s-1}^* i_s}(t+1) \geq q_{\min} \\
& \left(p_{j_1^* j_2^* \dots j_{s-1}^* i_s}(t) - \frac{\varepsilon}{2(r-1)} \right) \\
& \geq q_{\min} + \frac{\varepsilon}{2(r-1)} - \frac{\varepsilon}{2(r-1)} = q_{\min}. \tag{A.6}
\end{aligned}$$

This means that there exists at least two components in the set $A_s(t)$. Hence, using (9), we can derive the following inequality from the first term of the right-hand side of (A.3):

$$\begin{aligned}
& E\{\Delta p_{j_1^* j_2^* \dots j_s^*}(t) | p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B \wedge j_s^* \in A_s(t)\} \\
&= E\left\{ v_{j_1^* j_2^* \dots j_s^*}(t) - \frac{1}{|A_s(t)|} \sum_{l_s \in A_s(t)} v_{j_1^* j_2^* \dots j_{s-1}^* l_s}(t) \right\} \\
&= E\{v_{j_1^* j_2^* \dots j_s^*}(t)\} - \frac{1}{|A_s(t)|} \sum_{l_s \in A_s(t)} E\{v_{j_1^* j_2^* \dots j_{s-1}^* l_s}(t)\} \\
&> 0. \tag{A.7}
\end{aligned}$$

(The last inequality is derived from Lemma 1.)

Now, let us look at the second term $\Pr(j_s^* \in A_s(t))$. Since each action of the automaton $Z_{j_1^* j_2^* \dots j_{s-1}^*}$ is chosen by the probability larger than $q_{\min}$ ($p_{j_1^* j_2^* \dots j_{s-1}^* i_s}(t) \geq q_{\min}$ ($i_s = 1, 2, \dots, r$)), there is a nonzero probability ($\geq (q_{\min})^r$) that each of the r actions will have been chosen within r steps. Since $E\{v_{j_1^* j_2^* \dots j_s^*}(t)\} > E\{v_{j_1^* j_2^* \dots j_{s-1}^* i_s}(t)\}$ ($i_s \neq j_s^*$) from Lemma 1, we can get $\Pr\{v_{j_1^* j_2^* \dots j_s^*}(t+r) > v_{j_1^* j_2^* \dots j_{s-1}^* i_s}(t+r) \text{ for } \forall i_s \neq j_s^*\} > 0$ for $\forall t$. Hence, we can obtain

$$\Pr(j_s^* \in A_s(t)) > 0. \tag{A.8}$$

Therefore, from (A.7) and (A.8), we can prove Lemma A1.

Now, let us examine (A.1). Then, we can get the following relation concerning the action probability $p_{j_1^* j_2^* \dots j_s^*}(t)$ of the automaton $Z_{j_1^* j_2^* \dots j_{s-1}^*}$ ($s = 1, 2, \dots, N$) connected to the optimal path:

$$\begin{aligned}
& b_s(t) = E\{L^2(p_{j_1^* j_2^* \dots j_s^*}(t))\} \\
&= E\{L^2(p_{j_1^* j_2^* \dots j_s^*}(t)) | p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B\} \\
&\quad \times \Pr(p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B) \\
&\quad + E\{L^2(p_{j_1^* j_2^* \dots j_s^*}(t)) | p_{j_1^* j_2^* \dots j_s^*}(t) \in I_C\} \\
&\quad \times \Pr(p_{j_1^* j_2^* \dots j_s^*}(t) \in I_C) \\
&= E\{L^2(p_{j_1^* j_2^* \dots j_s^*}(t)) | p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B\} \\
&\quad \times \Pr(p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B). \tag{A.9}
\end{aligned}$$

By expanding $E\{L^2(p_{j_1^* j_2^* \dots j_s^*}(t+1))\}$, we can get

$$\begin{aligned}
& b_s(t+1) = E\{L^2(p_{j_1^* j_2^* \dots j_s^*}(t+1))\} \\
&= E\{L^2(p_{j_1^* j_2^* \dots j_s^*}(t+1)) | p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B\} \\
&\quad \times \Pr(p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B) \\
&\quad + E\{L^2(p_{j_1^* j_2^* \dots j_s^*}(t+1)) | p_{j_1^* j_2^* \dots j_s^*}(t) \in I_C\} \\
&\quad \times \Pr(p_{j_1^* j_2^* \dots j_s^*}(t) \in I_C). \tag{A.10}
\end{aligned}$$

By utilizing (8), (9), and (A.1), we can derive the following equality concerning the first term of the right-hand side:

$$\begin{aligned}
& E\{L^2(p_{j_1^* j_2^* \dots j_s^*}(t+1)) | p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B\} \\
&= E\left\{ \left(q_{\max} - \frac{\varepsilon}{2} - p_{j_1^* j_2^* \dots j_s^*}(t) \right. \right. \\
&\quad \left. \left. - \lambda_{j_1^* j_2^* \dots j_{s-1}^*}(t) \Delta p_{j_1^* j_2^* \dots j_s^*}(t) \right)^2 \middle| p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B \right\} \\
&= E\left\{ \left(q_{\max} - \frac{\varepsilon}{2} - p_{j_1^* j_2^* \dots j_s^*}(t) \right)^2 \middle| p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B \right\} \\
&\quad - 2\lambda_{j_1^* j_2^* \dots j_{s-1}^*}(t) E\left\{ \left(q_{\max} - \frac{\varepsilon}{2} - p_{j_1^* j_2^* \dots j_s^*}(t) \right) \right. \\
&\quad \left. \times \Delta p_{j_1^* j_2^* \dots j_s^*}(t) \middle| p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B \right\} \\
&\quad + \lambda_{j_1^* j_2^* \dots j_{s-1}^*}^2(t) E\{\Delta^2 p_{j_1^* j_2^* \dots j_s^*}(t) | p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B\} \\
&= E\{L^2(p_{j_1^* j_2^* \dots j_s^*}(t)) | p_{j_1^* j_2^* \dots j_s^*}(t) \in I_B\} \\
&\quad - 2\lambda_{j_1^* j_2^* \dots j_{s-1}^*}(t) d_s(t) + \lambda_{j_1^* j_2^* \dots j_{s-1}^*}^2(t) e_s(t). \tag{A.11}
\end{aligned}$$

Hence, from (A.10) and (A.11), we can get

$$b_s(t+1) = f_s(t)P_{sC}(t) + b_s(t) + P_{sB}(t)(\lambda_{j_1^*j_2^* \dots j_{s-1}^*}^2(t)e_s(t) - 2\lambda_{j_1^*j_2^* \dots j_{s-1}^*}(t)d_s(t)). \quad (\text{A.12})$$

From the above equation, we can derive

$$2 \sum_{k=1}^t P_{sB}(k)\lambda_{j_1^*j_2^* \dots j_{s-1}^*}(k)d_s(k) = b_s(1) - b_s(t+1) + \sum_{k=1}^t \times f_s(k)P_{sC}(k) + \sum_{k=1}^t P_{sB}(k)\lambda_{j_1^*j_2^* \dots j_{s-1}^*}^2(k)e_s(k). \quad (\text{A.13})$$

Since $b_s(t) \geq 0$ for an arbitrary time t , we can get from (A.13)

$$2 \sum_{k=1}^{\infty} P_{sB}(k)\lambda_{j_1^*j_2^* \dots j_{s-1}^*}(k)d_s(k) \leq b_s(1) + \sum_{k=1}^{\infty} f_s(k)P_{sC}(k) + \sum_{k=1}^{\infty} P_{sB}(k)\lambda_{j_1^*j_2^* \dots j_{s-1}^*}^2(k)e_s(k). \quad (\text{A.14})$$

Furthermore, $\lambda_{j_1^*j_2^* \dots j_{s-1}^*}^2(k)e_s(k) \leq \lambda_{j_1^*j_2^* \dots j_{s-1}^*}^2(k)(\overline{\Delta p})^2$ and $f_s(k) \leq \lambda_{j_1^*j_2^* \dots j_{s-1}^*}^2(k)(\Delta \bar{p})^2$. Therefore, it can be easily shown from (19) that the two infinite series of the right-hand side of (A.14) converge. Hence, $\sum_{k=1}^{\infty} P_{sB}(k)\lambda_{j_1^*j_2^* \dots j_{s-1}^*}(k)d_s(k)$ is bounded above.

Now that we have already proven Lemma A1, let us choose one $\bar{T}$ which satisfies (A.2). Then, from the fact that $\lambda_{j_1^*j_2^* \dots j_{s-1}^*}(t)d_s(t) \geq 0$ for an arbitrary time $t(t > \bar{T})$, it can be easily shown that $\sum_{k=\bar{T}}^{\infty} P_{sB}(k)\lambda_{j_1^*j_2^* \dots j_{s-1}^*}(k)d_s(k)$ is bounded above.

Hence, we can arrive at the following:

$\sum_{k=1}^{\infty} P_{sB}(k)\lambda_{j_1^*j_2^* \dots j_{s-1}^*}(k)d_s(k)$ exists and has a finite value.

By following the same procedure as in [20, p. 397], we can obtain (20). Q.E.D.

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