Parabolic Equations with Free Boundary Conditions

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§ 1. Introduction.

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\Gamma$. We are interested in the following problem

\[
\begin{aligned}
\frac{\partial u}{\partial t}(x, t) &= \Delta u(x, t) + g(u(x, t)) \quad \text{in } \Omega \times (0, \infty), \\
u(x, 0) &= C(t) \quad \text{(unknown function depending only on } t) \quad \text{on } \Gamma \times (0, \infty), \\
\int_{\Gamma} \frac{\partial u}{\partial \nu}(x, t) d\Gamma &= I(t) \quad \text{(given function)} \quad \text{in } (0, \infty), \\
u(x, 0) &= a(x) \quad \text{(given function)} \quad \text{in } \Omega,
\end{aligned}
\]

where $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ and $\frac{\partial}{\partial \nu}$ is the outward normal derivative to $\Gamma$. The function $g$ is uniformly Lipschitz continuous on $\mathbb{R}$.

The boundary conditions in our problem (P) are regarded as free boundary conditions in the following sense: values of $u$ on $\Gamma$ are not prescribed except that they are independent of $x \in \Gamma$. Such boundary conditions appear, for example, in the model problems describing the equilibrium of a confined plasma:

\[
\begin{aligned}
\Delta u &= \lambda u^-, \quad \text{in } \Omega, \\
u &= \text{(unknown) constant} \quad \text{on } \Gamma, \\
\int_{\Gamma} \frac{\partial u}{\partial \nu} d\Gamma &= I,
\end{aligned}
\]

where $\lambda$ and $I$ are given positive constants and $u^- = -\min \{0, u\}$. There are many contributions to the solvability of this problem (see, e.g., Berestycki and Brezis [1] and Temam [5]). We remark here that our problem (P) is not necessarily related with non-stationary problems in plasma physics. Our concern is to study (P) from a mathematical point of view.

The main purpose of this paper is to show the existence of a global solution of (P) in an elementary and constructive manner. In order to do so, we adopt a semi-
discretization method. This method has the advantage that unknown values of \( u \) on \( \Gamma' \) are explicitly determined as limits of suitable approximate functions.

The content of this paper is as follows. In \( \S 2 \) we prepare some notation and lemmas. \( \S 3 \) is devoted to the study of the existence and uniqueness of solutions for (P) in the framework of \( L^2 \)-theory. In \( \S 4 \) we establish regularity results for (P) in the classes of \( L^2 \)-spaces, \( L^p \)-spaces and Hölder spaces. As a particular result, it is proved that (P) has a unique classical solution if given functions \( a \) and \( f \) satisfy some appropriate conditions. In \( \S 5 \) we briefly state another approach to (P), which is based on analytic semi-group theory.

\section{Preliminaries.}

In this section we will prepare some notation and lemmas.

Let \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) denote the inner product and the norm in \( L^2(\Omega) \) or \( \{ L^2(\Omega) \}^d \). For every integer \( k \geq 1 \), \( H^k(\Omega) \) denotes the usual Sobolev space of order \( k \): the space of functions \( u \) such that \( u \) and its distributional derivatives up to order \( k \) belong to \( L^2(\Omega) \). We introduce the following closed subspace of \( H^1(\Omega) \)

\[ E = \{ u \in H^1(\Omega); \ iu = \text{constant} \}, \]

where \( i \) denotes the trace operator on \( \Gamma \).

The following lemma will be frequently used.

\textbf{Lemma 2.1.} \ For every \( \varepsilon > 0, \) there exists a positive number \( K_\varepsilon \) satisfying

\[
|i u| \leq \varepsilon \| Fu \| + K_\varepsilon \| u \| \quad \text{for all } u \in E,
\]

where \( Fu = (\partial u/\partial x_1, \partial u/\partial x_2, \ldots, \partial u/\partial x_d) \).

\textit{Proof.} We observe that for \( u \in E \)

\[
|i u| = \left( \int_{\Gamma} (i u)(x) d\Gamma / \text{meas} (\Gamma) \right) \leq \| i u \|_{L^2(\Gamma)} / \{ \text{meas} (\Gamma) \}^{1/2}.
\]

The result of Lions and Magenes [3, Theorem 9.4 in Chapter 1] yields

\[
\| i u \|_{L^2(\Gamma)} \leq C_\gamma \| u \|_{H^{1+\gamma}(\Omega)} \quad \text{for } u \in H^1(\Omega),
\]

where \( \gamma \) is an arbitrary number satisfying \( 0 < \gamma < 1/2 \), \( C_\gamma \) is a positive number depending on \( \gamma \) and \( H^{1+\gamma}(\Omega) = [H^1(\Omega), L^2(\Omega)]_{1/2 - \gamma} \). Moreover, using the interpolation result [3, Proposition 2.3 in Chapter 1], one can see that for every \( 0 < \gamma < 1/2 \) and \( u \in H^1(\Omega) \)

\[
\| i u \|_{L^2(\Omega)} \leq C'_\gamma \| u \|^{1/2 - \gamma} \| u \|^{1/2 + \gamma}_{H^1(\Omega)},
\]

with some \( C'_\gamma > 0 \). Therefore, (2.1) will be derived from (2.2) and (2.3). q.e.d.
Finally we state Gronwall’s inequality of discrete type.

Lemma 2.2. Let \( \{a_n\}_{n=0}^{\infty} \) be a sequence of positive numbers such that

\[
(2.4) \quad a_j \leq b + \sum_{i=0}^{j-1} b_i a_i, \quad j=0, 1, 2, \ldots,
\]

with \( b \geq 0 \) and \( 1 > \alpha \geq b_j \geq 0 \) for every \( j \geq 0 \). Then

\[
(2.5) \quad a_j \leq \frac{1}{1 - \alpha} (b + b_0 a_0) \exp \left( \frac{1}{1 - \alpha} \sum_{i=1}^{j-1} b_i \right), \quad j=1, 2, \ldots,
\]

with the understanding that \( \sum_{i=1}^{j-1} b_i = 0 \).

Proof. It follows from (2.4) that

\[ (1 - \alpha) a_j \leq (1 - b_j) a_j \leq b + \sum_{i=0}^{j-1} b_i a_i; \]

so that

\[
(2.6) \quad a_j \leq b' + \sum_{i=0}^{j-1} b_i a_i,
\]

with \( b' = b/(1 - \alpha) \) and \( b' = b_0/(1 - \alpha) \). If we denote by \( f_{j-1} \) the right-hand side of (2.6), then

\[ f_j - f_{j-1} = b' a_j \leq b' f_{j-1}, \]

from which we deduce

\[
(2.7) \quad f_j \leq (1 + b') f_{j-1} \leq \cdots \leq \prod_{i=1}^{j} (1 + b'_i) \leq (b' + b'_0 \exp \left( \sum_{i=1}^{j-1} b'_i \right). \]

Since \( a_j \leq f_{j-1} \) for every \( j \geq 1 \), one obtains (2.5) from (2.7).

\( \text{q.e.d.} \)

§ 3. Existence and uniqueness.

In this section we will establish the existence and uniqueness result for \((P)\). Hereafter, we denote by \( L \) the Lipschitz constant of \( g \) and assume, for simplicity, \( g(0) = 0 \). Given functions \( a \) and \( I \) are assumed to satisfy

\[
(3.1) \quad a \in E,
\]

\[
(3.2) \quad I \in C^'[0, \infty).
\]

Our result is

Theorem 3.1. There exists a unique function \( u \) such that
\( u \in C([0, T]; L^2(\Omega)) \cap L^\infty(0, T; E) \cap L^r(0, T; H^r(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega)) \)

for every \( T > 0 \) and

\[
\frac{\partial u}{\partial t} = Au + g(u) \quad \text{a.e. in } \Omega \times (0, \infty),
\]

(3.3)

\[
u = C(t) \text{ (independent of } x) \quad \text{a.e. on } \Gamma \times (0, \infty),
\]

(3.4)

\[
\int_{\Gamma} \frac{\partial u}{\partial \nu} \, d\Gamma = I(t) \quad \text{a.e. in } (0, \infty),
\]

(3.5)

\[
u(\cdot, 0) = a \quad \text{a.e. in } \Omega.
\]

(3.6)

**Remark 3.1.** We say that \( u \) is a strong solution of (P) if \( u \) has the properties stated in Theorem 3.1.

We will prove Theorem 3.1 by semi-discretization method; the discretization is carried out only in the time variable (see, e.g., Temam [6, Chapter 3, §§ 4–5]).

Let \( T > 0 \) be fixed and divide \([0, T]\) into \( n \) intervals of equal length \( \tau = T/n \). We define recursively a family \( \{u^n_i\}_{i=0}^n \), where \( u^n_i \) will approximate a solution of (P) on \(((i-1)\tau, i\tau)\).

We begin with \( u^n_0 = a \); then, when \( u^n_0, u^n_1, \ldots, u^n_{i-1} \) are known, we define \( u^n_i \) as a solution of

\[
\frac{u^n_i - u^n_{i-1}}{\tau} = Au^n_i + g(u^n_{i-1}) \quad \text{in } \Omega,
\]

(3.7)

\[
u^n_i = C^n_i \text{ (unknown constant) on } \Gamma,
\]

(3.8)

\[
\int_{\Gamma} \frac{\partial u^n_i}{\partial \nu} \, d\Gamma = I^n_i \equiv I(i\tau).
\]

(3.9)

**Lemma 3.2.** A family \( \{u^n_i\}_{i=1}^n \subset E \cap H^r(\Omega) \) is uniquely determined from (3.7)–(3.9).

**Proof.** Assume that \( u^n_j \in E, j = 0, 1, \ldots, i - 1 \), are known. If we set \( \nu^n_i = u^n_i - C^n_i \), then our original problem (3.7)–(3.9) is reduced to finding a pair of \( \nu^n_i \) and \( C^n_i \) satisfying

\[
\begin{aligned}
(I-\tau A)\nu^n_i &= h^n_{i-1} - C^n_i \quad \text{in } \Omega, \\
\nu^n_i &= 0 \quad \text{on } \Gamma, \\
\int_{\Gamma} \frac{\partial \nu^n_i}{\partial \nu} \, d\Gamma &= I^n_i,
\end{aligned}
\]

(3.10)

where \( h^n_{i-1} = u^{i-1}_n + \tau g(u^{i-1}_n) \in L^r(\Omega) \).
It is convenient to introduce a closed linear operator \( \Delta_D \) (in \( L^2(\Omega) \)) defined by

\[-\Delta_D u = -\Delta u \quad \text{for } u \in D(\Delta_D) = H^3_0(\Omega) \cap H^2(\Omega),\]

where \( H^3_0(\Omega) \) denotes the closed subspace of \( H^4(\Omega) \) of functions vanishing on \( \Gamma \). Since \( v_n^i \) is expressed in terms of \( \Delta_D \) by

\[ v_n^i = (I - \tau \Delta_D)^{-1}(h_n^{i-1} - C_n^i), \tag{3.11} \]

it suffices to determine \( C_n^i \) to solve (3.10). Substituting (3.11) into the last equation in (3.10) leads to

\[ \int_{\Gamma} \frac{\partial}{\partial \nu}(I - \tau \Delta_D)^{-1} \cdot 1 d\Gamma \cdot C_n^i = -I_n^i + \int_{\Gamma} \frac{\partial}{\partial \nu}(I - \tau \Delta_D)^{-1} h_n^{i-1} d\Gamma. \tag{3.12} \]

The strong maximum principle for elliptic equations assures that the coefficient of \( C_n^i \) in (3.10) is positive; so that \( C_n^i \) is uniquely determined from (3.12) and \( v_n^i \in H^3_0(\Omega) \cap H^2(\Omega) \) is given by (3.11). Thus the proof is complete. q.e.d.

In the following two lemmas, we will state various estimates for \( u_n^i \).

**Lemma 3.3.**

(i) \( \max \{ ||u_n^i||; 0 \leq j \leq n \} \) is bounded independently of \( n \).

(ii) The sum \( \tau \sum_{i=1}^n ||F u_n^i||^2 \) is bounded independently of \( n \).

**Proof.** For notational convenience, we simply write \( u^i, C^i \) and \( I^i \) in place of \( u_n^i, C_n^i \) and \( I_n^i \).

Multiplying (3.7) by \( u^i \) and integrating over \( \Omega \) we get by the use of Green's formula

\[ ||u^i||^2 - (u^{i-1}, u^i) = \tau \int_{\Gamma} \frac{\partial u^i}{\partial \nu} u^i d\Gamma - \tau ||F u^i||^2 + \tau (g(u^{i-1}), u^i). \tag{3.13} \]

The left-hand side of (3.13) is bounded from below by

\[ \frac{1}{2} ||u^i||^2 - ||u^{i-1}||^2. \]

Since \( u^i \in \mathcal{E} \), it follows from (3.8), (3.9) and Lemma 2.1 (with \( \varepsilon = 1 \)) that

\[ \int_{\Gamma} \frac{\partial u^i}{\partial \nu} u^i d\Gamma = |C^i I^i| \leq \sup_{\varepsilon \in \mathcal{E}} |I(\varepsilon)||F u^i|| + K_1 ||u^i|| \]

\[ \leq \frac{1}{4} ||F u^i||^2 + M (||u^i||^2 + 1), \tag{3.14} \]

where \( M \) is a positive constant independent of \( n \). By the Lipschitz continuity of \( g \),

\[ |(g(u^{i-1}), u^i)| \leq L ||u^{i-1}|| ||u^i|| \leq \frac{L}{2} (||u^{i-1}||^2 + ||u^i||^2). \]
Using these results we rearrange (3.13) to get
\begin{equation}
\|u^i\|^2 - \|u^i - u^{i-1}\|^2 + \tau \|\nabla u^i\|^2 \leq \tau M'(1 + \|u^{i-1}\|^2 + \|u^i\|^2),
\end{equation}
with some $M' > 0$ independent of $n$. Summation of (3.15) for $i = 1, 2, \ldots, j$, yields
\begin{equation}
\|u^i\|^2 + \tau \sum_{i=1}^{j-1} \|\nabla u^i\|^2 \leq \|a\|^2 + M' \tau \sum_{i=1}^{j-1} (1 + \|u^{i-1}\|^2 + \|u^i\|^2)
\end{equation}
\begin{equation}
\leq \|a\|^2 + M'T + 2M' \tau \sum_{i=1}^{j} \|u^i\|^2.
\end{equation}
Hence applying Lemma 2.2 to (3.16) we conclude the proofs of (i) and (ii). q.e.d.

**Lemma 3.4.**
(i) \( \max \{\|F u_n\|: 0 \leq j \leq n\} \) is bounded independently of $n$.
(ii) The sum \( \tau \sum_{j=1}^{n} \|\Delta u_j\|^2 \) is bounded independently of $n$.
(iii) The sum \( \tau \sum_{j=1}^{n} \|(u_j - u_{j-1})/\epsilon\|^2 \) is bounded independently of $n$.

**Proof.** As in the proof of Lemma 3.3, we drop the subscript $n$. Multiplying (3.7) by \(-\Delta u^i\) and integrating over $\Omega$ one obtains
\begin{equation}
-(u^i, \Delta u^i) + (u^{i-1}, \Delta u^i) = -\tau \|\Delta u^i\|^2 - \tau (g(u^{i-1}), \Delta u^i)
\end{equation}
\begin{equation}
\leq -\tau \|\Delta u^i\|^2 + \tau L \|u^{i-1}\| \|\Delta u^i\|
\leq -\frac{\tau}{2} \|\Delta u^i\|^2 + \frac{1}{2} \tau L^2 \|u^{i-1}\|^2.
\end{equation}
By (3.8), (3.9) and Green's formula,
\begin{equation}
-(u^i, \Delta u^i) + (u^{i-1}, \Delta u^i) = -\int_{\Gamma} u^i \frac{\partial u^i}{\partial v} d\Gamma + \|\nabla u^i\|^2 + \int_{\Gamma} u^{i-1} \frac{\partial u^i}{\partial v} d\Gamma - (F u^{i-1}, F u^i)
\end{equation}
\begin{equation}
= -C^i T + C^{i-1} T + \frac{1}{2} \{\|F u^i\|^2 - \|F u^{i-1}\|^2 + \|F(u^i - u^{i-1})\|^2 \}.
\end{equation}
Using this fact we rearrange (3.17);
\begin{equation}
\|F u^i\|^2 - \|F u^{i-1}\|^2 + \tau \|\Delta u^i\|^2 \leq \tau L^2 \|u^{i-1}\|^2 + 2\{C^i T - C^{i-1} T\}.
\end{equation}
Summing (3.18) for $i = 1, 2, \ldots, j$ leads to
\begin{equation}
\|F u^j\|^2 + \tau \sum_{i=1}^{j} \|\Delta u^i\|^2 \leq \|\nabla a\|^2 + \tau L^2 \sum_{i=1}^{j-1} \|u^i\|^2 + 2 \sum_{i=1}^{j} \{C^i T - C^{i-1} T\}
\end{equation}
\begin{equation}
= \|\nabla a\|^2 + \tau L^2 \sum_{i=1}^{j-1} \|u^i\|^2 + 2\{C^j T - C^1 T + \sum_{i=1}^{j} C^i (T - T^{i-1})\}
\end{equation}
\begin{equation}
\leq N + \frac{1}{2} \|\nabla u^j\|^2 + 2 \sup_{0 \leq t \leq T} \|\dot{a}(t)\| \sum_{i=1}^{j} |C^i|,
\end{equation}
where $N$ is a positive constant independent of $n$ and \( \cdot \) means $d/dt$. 

(In the last inequality of (3.19), we have used (3.14) and Lemma 3.3 (i).) Since Lemma 2.1 (with \( \varepsilon = 1 \)) implies
\[
\tau \sum_{i=1}^{n} |C^i| \leq \tau \sum_{i=1}^{n} (||Fu^i|| + K_i ||u^i||) \leq \tau \sum_{i=1}^{n} (||Fu^i||^2 + 1 + K_i ||u^i||),
\]
it is seen from Lemma 3.3 that the sum \( \tau \sum_{i=1}^{n} |C^i| \) is bounded independently of \( n \). Consequently, the assertions (i) and (ii) are derived from (3.19). Moreover, it suffices to use (3.7) to prove (iii).
q.e.d.

For each fixed \( n \geq 1 \), we associate to the family \( \{u_n^i\}_{i=0}^{n} \) the following approximate functions:
\[
u_n(t) = u_n^i \quad \text{for } t \in ((i-1)\tau, i\tau], \quad i = 0, 1, 2, \ldots, n,
\]
and
\[
u_n(t) = u_n^{i-1} + (u_n^i - u_n^{i-1})(t - (i-1)\tau)/\tau \quad \text{for } t \in [(i-1)\tau, i\tau], \quad i = 1, 2, \ldots, n.
\]
Then our discretized problem (3.7)-(3.9) is written, in terms of \( u_n \) and \( v_n \), as follows:

(3.20) \[ \frac{\partial v_n}{\partial t} = \Delta u_n + g(u_n(t-\tau)) \quad \text{in } \Omega \times [0, T] \quad (t \neq i\tau \text{ for } i = 0, 1, \ldots, n), \]

(3.21) \[ v_n = C_n(t) \quad \text{on } \Gamma \times [0, T], \]

(3.22) \[ \int_\Gamma \frac{\partial v_n}{\partial \nu} \, d\Gamma = I_n(t) \quad \text{in } [0, T], \]

where \( C_n \) and \( I_n \) are defined by
\[
C_n(t) = C_n^{i-1} + (C_n^i - C_n^{i-1})(t - (i-1)\tau)/\tau \quad \text{and} \quad I_n(t) = I_n^{i-1} + (I_n^i - I_n^{i-1})(t - (i-1)\tau)/\tau \quad \text{for } t \in [(i-1)\tau, i\tau], \quad i = 1, 2, \ldots, n.
\]

The approximate functions \( u_n \) and \( v_n \) have the following properties.

**Lemma 3.5.**

(i) \( \{u_n\}_{n=1}^{\infty} \) is bounded in \( L^\infty(0, T; E) \cap L^2(0, T; H^4(\Omega)) \).

(ii) \( \{v_n\}_{n=1}^{\infty} \) is bounded in \( C([0, T]; E) \cap L^2(0, T; H^4(\Omega)) \) and \( \{\partial v_n/\partial t\}_{n=1}^{\infty} \) is bounded in \( L^2(0, T; L^2(\Omega)) \).

(iii) \( \lim_{n \to \infty} (u_n - v_n) = 0 \) in \( L^2(0, T; L^2(\Omega)) \).

**Proof.** From the definition of \( u_n \) and \( v_n \), the assertions (i) and (ii) are direct consequences of Lemmas 3.3 and 3.4.

To prove (iii), we use
\[
\int_0^T \|u_n(t) - v_n(t)\|^2 \, dt = \sum_{i=1}^{n} \int_{(i-1)\tau}^{i\tau} \|u_n^i - u_n^{i-1}\|^2 \, (t - i\tau)^{\varepsilon} \, dt
\]
\[
= \frac{1}{3} \varepsilon \left\{ \tau \sum_{i=1}^{n} \|u_n^i - u_n^{i-1}\|/\tau \right\},
\]
where \( \varepsilon = T/n \). Hence, Lemma 3.4 (iii) enables us to get the conclusion. q.e.d.
We will construct a strong solution of (P) by compactness argument. From Lemma 3.5 (i), (ii), it is possible to extract from \(\{u_n\}\) and \(\{v_n\}\) two subsequences \(\{u_{n'}\}\) and \(\{v_{n'}\}\) such that

\[
\begin{align*}
u_{n'} & \rightharpoonup u \quad \text{in } L^2(0, T; H^1(\Omega)) \text{ weakly,} \\
v_{n'} & \rightharpoonup v \quad \text{in } L^\infty(0, T; E) \text{ weakly-star,}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u_{n'}}{\partial t} & \rightharpoonup \frac{\partial v}{\partial t} \quad \text{in } L^2(0, T; L^2(\Omega)) \text{ weakly,}
\end{align*}
\]

as \(n' \to \infty\). Moreover, Lemma 3.5 (iii) together with (3.23) and (3.24) implies \(u = v\).

We want to show that \(u\) is a strong solution of (P). Since we have to prove \(g(u_{n'}(t - \tau')) - g(u(t))\), with \(\tau' = T/n'\), in an appropriate sense, we need a strong convergence result for \(\{u_{n'}\}\) or \(\{v_{n'}\}\). By Lemma 3.5 (ii), \(v_{n'}\) are uniformly bounded in \(C([0, T]; E)\) and \(v_n: [0, T] \to L^2(\Omega)\) are equicontinuous. Therefore, since \(E\) is compactly embedded in \(L^2(\Omega)\), Ascoli-Arzelà's Theorem implies

\[
\begin{align*}
u_{n'} & \longrightarrow u \quad \text{in } C([0, T]; L^2(\Omega)) \quad \text{as } n' \to \infty.
\end{align*}
\]

By the Lipschitz continuity of \(g\),

\[
|g(u_{n'}(t - \tau')) - g(u(t))| \leq L\left(|u_{n'}(t - \tau') - v_{n'}(t - \tau')| + |v_{n'}(t - \tau') - u(t)| + |v_{n'}(t) - u(t)|\right);
\]

so that we see from Lemma 3.5 (iii) and (3.26) that

\[
\lim_{n' \to \infty} g(u_{n'}(t - \tau')) = g(u) \quad \text{in } L^2(0, T; L^2(\Omega)).
\]

Now we are ready to prove that \(u\) is a strong solution of (P). Setting \(n = n'\) in (3.20) and letting \(n' \to \infty\), we find that (3.3) holds in the sense \(L^2(0, T; L^2(\Omega))\) for every \(T > 0\) because convergence properties (3.23), (3.25) (set \(v = u\)) and (3.27) are already established.

It is easy to see that \(u\) satisfies the initial condition (3.6).

We will verify (3.4). It follows from Lemma 2.1 and (3.21) that for every \(\varepsilon > 0\)

\[
|C_{\nu'}(t) - C_{\nu}(t)| \leq \varepsilon \left\| F(v_{n'}(t) - v_{n'}(t)) \right\| + K \| v_{n'}(t) - v_{n'}(t) \|.
\]

Therefore, in view of Lemma 3.5 (ii) and (3.26), we find that \(C_{\nu}(t)\) converge to a continuous function \(C(t)\) uniformly on \([0, T]\). Similarly, recalling the trace estimate (2.3) one can deduce that \(\tilde{\tau} v_{n'}(t) = C_{\nu}(t)\) converge to \(\tilde{\tau} u(t)\) in \(L^2(\Omega)\) for a.e. \(t \in [0, T]\). Hence, (3.4) has been proved.

The other boundary condition (3.5) is verified as follows. Interpret the integral
\[ \int_R \frac{\partial u}{\partial v} d\Gamma = (f, u)_{H^1(\Omega)} \quad \text{for } u \in H^1(\Omega). \]

Since \((f, v_n(t))_{H^1(\Omega)} = I_n(t)\) (by (3.22)) and \(v_n\) converge to \(u\) weakly in \(L^2(0, T; H^1(\Omega))\), we get, after letting \(n' \to \infty\),

\[ \int_R \frac{\partial u}{\partial v} (t) d\Gamma = (f, u(t)) = I(t) \quad \text{for a.e. } t \in [0, T]. \]

Thus the existence of a strong solution of (P) has been proved.

It remains to show the uniqueness. Let \(v\) be another strong solution of (P). If we set \(w = u - v\), then

\[
\begin{aligned}
\frac{\partial w}{\partial t} &= \Delta w + (g(u) - g(v)) &\text{a.e. in } \Omega \times (0, \infty), \\
w &= C(t) \text{ (independent of } x) &\text{a.e. on } \Gamma \times (0, \infty), \\
\int_R \frac{\partial w}{\partial v} d\Gamma &= 0 &\text{a.e. in } (0, \infty), \\
w(\cdot, 0) &= 0 &\text{a.e. in } \Omega.
\end{aligned}
\]

Multiplying the first equation of (3.28) by \(w\) and integrating over \(\Omega\), we have

\[ \frac{1}{2} \frac{d}{dt} \| w(t) \|^2 + \| Fw(t) \|^2 = (g(u(t)) - g(v(t))), w(t)) \leq L \| w(t) \|^2. \]

Integrating (3.29) over \((0, t)\) gives

\[ \| w(t) \|^2 \leq 2L \int_0^t \| w(s) \|^2 ds, \]

which implies \(w = 0\) (i.e., \(u = v\)) by Gronwall’s inequality.

\textbf{q.e.d.}

\textbf{Remark 3.2.} In the proof of Theorem 3.1, a strong solution of (P) has been approximated by a suitable subsequence \(\{u_n\}\) (or \(\{v_n\}\)) of \(\{u_n\}\) (or \(\{v_n\}\)). However, the uniqueness result assures that the sequence \(\{u_n\}\) (or \(\{v_n\}\)) itself converges to the strong solution.

\textbf{§ 4. Regularity.}

We first study regularity properties of the strong solution of (P) within the framework of \(L^2\)-theory. It is convenient to introduce the following function space
\[ V = \left\{ u: \Omega \times [0, \infty) \to R; \text{ess sup}_{0 \leq t < \infty} \| u(t) \| + \int_0^T \| F_u(s) \| \, ds < \infty \text{ for every } T > 0 \right\}. \]

(By Theorem 3.1, the strong solution \( u \) satisfies, \( F_u \in V \).)

Throughout this section, the given functions \( a \) and \( I \) are assumed to satisfy, in addition to (3.1) and (3.2), the following compatibility condition

\begin{equation}
I(0) = \int_\Gamma r \frac{\partial a}{\partial \nu} \, d\Gamma.
\end{equation}

Our first result is

**Theorem 4.1.** Assume \( \Delta a \in E \) and \( I \in C^1[0, \infty) \). Then the strong solution \( u \) of (P) satisfies \( \partial u/\partial t \in E \) for a.e. \( t \in [0, \infty) \) and \( \partial u/\partial t, \partial^2 u/\partial x_i \partial t \in V \) for every \( i \).

**Remark 4.1.** Since Theorem 4.1 asserts \( \partial u/\partial t \in L^1(0, T; H^2(\Omega)) \), one can also see \( \partial^2 u/\partial t^2 \in L^1(0, T; L^2(\Omega)) \) by noting

\[ \partial^2 u/\partial t^2 = \Delta (\partial u/\partial t) + g'(u) \partial u/\partial t. \]

We will complete the proof of Theorem 4.1 by establishing the following four lemmas. The first is concerned with the continuity of \( u \) at \( t = 0 \).

**Lemma 4.2.** If \( \Delta a \in L^1(\Omega) \), then \( \sup_{0 < t < T} \| u(t) - a \| / t \) is bounded for every \( T > 0 \).

**Proof.** Let \( T > 0 \) be fixed. Setting \( v(t) = u(t) - a \), we have

\begin{equation}
\frac{1}{2} \frac{d}{dt} \| v(t) \|^2 = (\partial v(t)/\partial t, v(t))
\end{equation}

\[ = (\Delta v(t), v(t)) + (g(u(t))) + \Delta a, v(t))
\]

\[ = (C(t) - C(0)(I(t) - I(0)) - \| F v(t) \|^2 + (g(u(t)) + \Delta a, v(t)). \]

Lemma 2.1, together with \( u \in L^\infty(0, T; E) \) and (3.2), gives

\[ |C(t) - C(0)||I(t) - I(0)| \leq Mt, \quad 0 \leq t \leq T, \]

where \( M \) is a positive constant. Moreover,

\[ |(g(u(t)) + \Delta a, v(t))| \leq M' \| v(t) \|, \quad 0 \leq t \leq T, \]

with some \( M' > 0 \). Using these estimates we integrate (4.2) over \( (0, t) \) to get

\[ \| v(t) \|^2 \leq Mt^2 + 2M' \int_0^t \| v(s) \| \, ds, \quad 0 \leq t \leq T. \]

This inequality implies

\[ \| v(t) \| = \| u(t) - a \| \leq (\sqrt{M} + M')t, \quad 0 \leq t \leq T. \]

q.e.d.
Using Lemma 4.2 we will show a (spatial) regularity property of \( \partial u/\partial t \).

**Lemma 4.3.** If \( \Delta a \in L^2(\Omega) \) and \( I \in C^2[0, \infty) \), then \( \partial u(\cdot)/\partial t \in E \) for a.e. \( t \in [0, \infty) \) and \( \partial u/\partial t \in V \).

**Proof.** Let \( T > 0 \) be fixed. For each \( 0 < h \leq 1 \), we use the following notation

\[ u^h(t) = (u(t + h) - u(t))/h. \]

Since \( u^h \) satisfies

\[
\begin{align*}
\frac{\partial u^h}{\partial t} &= \Delta u^h + g(u)^h & \text{in } \Omega \times (0, T), \\
u^h &= C^h(t) & \text{on } \Gamma \times (0, T), \\
\int_{\Gamma} \frac{\partial u^h}{\partial \nu} \, d\Gamma &= I^h(t) & \text{in } (0, T),
\end{align*}
\]

we obtain by the use of Green’s formula

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| u^h(t) \|^2 &= -\| F u^h(t) \|^2 + C^h(t)I^h(t) + (g(u)^h(t), u^h(t)) \\
&\leq -\| F u^h(t) \|^2 + L \| u^h(t) \|^2 + C^h(t)I^h(t),
\end{align*}
\]

where the Lipschitz continuity of \( g \) has been used. Integrating (4.4) over \( (0, t) \) gives

\[
\begin{align*}
\| u^h(t) \|^2 + 2 \int_0^t \| F u^h(s) \|^2 \, ds &\leq \| u^h(0) \|^2 + 2L \int_0^t \| u^h(s) \|^2 \, ds \\
&+ 2 \int_0^t C^h(s)I^h(s) \, ds, \quad 0 \leq t \leq T.
\end{align*}
\]

By Lemma 4.2, the first term of the right-hand side of (4.5) is bounded independently of \( h \). The last term is expressed as

\[
\frac{2}{h} \int_0^{t+h} C(s)I^h(s-h) \, ds - \frac{2}{h} \int_0^h C(s)I^h(s) \, ds + 2 \int_0^t C(s)I^h(s-h) - I^h(s) \, ds.
\]

Here all the terms are bounded independently of \( h \) because \( C \in L^\infty(0, T) \) and \( I \in C^2[0, T] \). Consequently, (4.5) becomes

\[
\| u^h(t) \|^2 + 2 \int_0^t \| F u^h(s) \|^2 \, ds \leq M + 2L \int_0^t \| u^h(s) \|^2 \, ds, \quad 0 \leq t \leq T,
\]

with a positive constant \( M \) independent of \( h \). Applying Gronwall’s inequality to (4.6) yields

\[
\| u^h(t) \|^2 + 2 \int_0^t \| F u^h(s) \|^2 \, ds \leq Me^{2Lt}, \quad 0 \leq t \leq T,
\]

\[
\| u^h(t) \|^2 + 2 \int_0^t \| F u^h(s) \|^2 \, ds \leq Me^{2Lt}, \quad 0 \leq t \leq T,
\]
which assures $\partial u/\partial t \in V$ by letting $h \downarrow 0$.

It remains to show $\tau(\partial u(t)/\partial t)=\text{constant}$ for a.e. $t \in [0, T]$. For this purpose, we use the trace estimate (2.3) by putting $u^h - \partial u/\partial t$ in place of $u$. Since $\|u^h - \partial u/\partial t\|_{H^1(\Omega)}$ are uniformly bounded in $L^2(0, T)$ for all $0 < h \leq 1$ and $u^h$ converge to $\partial u/\partial t$ in $L^2(0, T; L^2(\Omega))$ as $h \downarrow 0$, we find

$$\tau(\partial u(t)/\partial t) = \hat{C}(=dC/\partial t) \quad \text{in } L^2(0, T; L^2(\Omega));$$

that is $\tau(\partial u(t)/\partial t) = \text{constant}$ for a.e. $t \in [0, T]$.

q.e.d.

Our third lemma is concerned with the continuity of $Fu$ at $t=0$.

**Lemma 4.4.** If $\Delta a \in E$, then $\sup_{0 < t \leq T} \{\|F(u(t) - a)/|t|\}$ is bounded for every $T > 0$.

**Proof.** The proof is similar to that of Lemma 4.2. Let $T > 0$ be fixed. Setting $v(t) = u(t) - a$, we can derive

$$\frac{1}{2} \frac{d}{dt} \|Fv(t)\|^2 = (F(\partial v(t)/\partial t), Fv(t))$$

(4.8)

$$= \int_F \frac{\partial v}{\partial t}(t) \frac{\partial v}{\partial t}(t) d\Gamma - \left(\frac{\partial v}{\partial t}(t), \Delta v(t)\right)$$

$$= \hat{C}(t)(I(t) - I(0)) - \|\Delta v(t)\|^2 - (g(u(t)) - g(a), \Delta v(t))$$

$$- (\Delta a + g(a), \Delta v(t)).$$

By the Lipschitz continuity of $g$ and Lemma 4.3,

$$|(g(u(t)) - g(a), \Delta v(t))| \leq \frac{1}{2} \|\Delta v(t)\|^2 + \frac{1}{2} L^2 t^2 \text{ ess sup}_{0 \leq t \leq T} \|\partial u(t)/\partial t\|^2.$$

Moreover $g(a)$ belongs to $E$ (see Treves [7, Lemma 28.1]); so that

$$-(\Delta a + g(a), \Delta v(t)) = - \tau(\Delta a + g(a))(I(t) - I(0)) + (F(\Delta a + g(a)), Fv(t))$$

$$\leq t \sup_{0 \leq s \leq t} |\hat{I}(t)| ||(\Delta a + g(a))|| ||Fv(t)||.$$

Using these results we integrate (4.8) over $(0, t)$ to obtain

$$\|Fv(t)\|^2 + \int_0^t \|\Delta v(s)\|^2 ds \leq 2(C(t) - C(0))(I(t) - I(0)) - 2 \int_0^t (C(s) - C(0))\hat{I}(s) ds$$

(4.9)

$$+ M(t^2 + \int_0^t \|Fv(s)\| ds), \quad 0 \leq t \leq T,$$

where $M$ is a positive constant. We evaluate the first two terms in the right-hand side of (4.9) by using Lemmas 2.1 and 4.2; the first is bounded by

$$2t \sup_{0 \leq t \leq T} |\hat{I}(t)||\|Fv(t)\| + K||v(t)||| \leq \frac{1}{2} \|Fv(t)\|^2 + M't^2, \quad 0 \leq t \leq T,$$
and the second is bounded by

\[ 2 \sup_{0 \leq s \leq t} |\dot{I}(s)| \int_0^t \left( \| Fv(s) \| + K_1 \| u(s) \| \right) ds \leq M'(t^2 + \int_0^t \| Fu(s) \| ds), \quad 0 \leq t \leq T, \]

where \( M' \) is a suitable positive constant. Therefore, we rearrange (4.9) to get

\[ \| Fu(t) \|^2 \leq M''(t^2 + \int_0^t \| Fu(s) \| ds), \quad 0 \leq t \leq T, \]

with some \( M'' > 0 \). This inequality yields the assertion of Lemma 4.4 (see the proof of Lemma 4.2).

Finally the proof of Theorem 4.1 will be completed if the following lemma is proved.

**Lemma 4.5.** Under the assumptions of Theorem 4.1, \( \partial u / \partial x, \partial t \in V \) for every \( i \).

**Proof.** The proof is similar to that of Lemma 4.3: we multiply the first equation of (4.3) by \( -\Delta u^h \) (in place of \( u^h \)). Then

\[ \frac{1}{2} \frac{d}{dt} \| Fu^h(t) \|^2 + \| \Delta u^h(t) \|^2 = \dot{C}^h(t) I^h(t) - (g(u^h(t), \Delta u^h(t)) \]

\[ \leq \dot{C}^h(t) I^h(t) + \frac{1}{2} \left( \| \Delta u^h(t) \|^2 + L^2 \| u^h(t) \|^2 \right), \]

for \( 0 \leq t \leq T \). By (4.7), integrating (4.10) over \((0, t)\) leads to

\[ \| Fu^h(t) \|^2 + \int_0^t \| \Delta u^h(s) \|^2 ds \leq M + 2 \int_0^t \dot{C}^h(s) I^h(s) ds, \quad 0 \leq t \leq T, \]

where \( M \) is a positive constant independent of \( h \). We will evaluate the last term in the right-hand side of (4.11), which is expressed as

\[ -2C^h(0) I^h(0) - 2 \int_0^t C^h(s) \dot{I}^h(s) ds + 2C^h(t) I^h(t). \]

Since Lemma 2.1 and (4.7) imply

\[ |C^h(t)| = |\dot{C}^h(t)| \leq \| Fu^h(t) \| + M', \quad 0 \leq t \leq T, \]

with a positive constant \( M' \) independent of \( h \), it is easy to see from Lemma 4.4 and (4.7) that the first two terms of (4.12) are bounded independently of \( h \). The last term is bounded by

\[ \frac{1}{2} \| Fu^h(t) \|^2 + M'', \]

where \( M'' \) is a positive constant independent of \( h \) (use (4.13)). Therefore, by going
back to (4.11), \( \{ Fu^k \} \) is bounded in \( L^\infty(0, T; \{ L^2(\Omega)^d \}) \cap L^2(0, T; \{ H^1(\Omega) \}) \). Consequently, letting \( h \downarrow 0 \), we conclude the proof. q.e.d.

We continue the study of regularity properties of \( u \) in \( L^p \)-spaces and Hölder spaces. It is convenient to employ some function spaces introduced by Ladyženskaja, Solonnikov and Ural'ceva [2]:

\[
W^{2,1}_p(\Omega), \quad W^{2,1}_p(\Omega \times (0, T)) \quad \text{with} \quad p \geq 2 \quad \text{and} \\
H^{2,1+\mu}_p(\overline{\Omega} \times [0, T]) \quad \text{with} \quad 0 < \mu < 1.
\]

We remark here that the inclusion relation

(4.14) \( W^{2,1}_p(\Omega \times (0, T)) \subseteq L^q(\Omega \times (0, T)) \)

holds if \( 1/q \geq 1/p - 2/(d+2) \) (see [2, p.80]).

**Proposition 4.6.** Under the assumptions of Theorem 4.1, the following statements hold true.

(i) If \( a \in W^{2,1}_p(\Omega) \), then \( u \in W^{2,1}_p(\Omega \times (0, T)) \) for every \( T > 0 \).

(ii) Suppose that second-order derivatives of \( a \in C^2(\overline{\Omega}) \) are Hölder continuous on \( \overline{\Omega} \). Then \( u \in H^{2,1+\mu}_p(\Omega \times [0, T]) \) with some \( 0 < \mu < 1/2 \) for every \( T > 0 \).

**Proof.** (i) Let \( T > 0 \) be fixed. If we put \( v = u - C \), then \( v \) satisfies

\[
\begin{align*}
\frac{\partial v}{\partial t} &= \Delta v + g(u) - \dot{C} \quad \text{in} \quad \Omega \times (0, T), \\
v &= 0 \quad \text{on} \quad \Gamma \times (0, T), \\
v(-, 0) &= a - \gamma a \quad \text{in} \quad \Omega.
\end{align*}
\]

Observe that

(4.16) \( \dot{C} \in L^\infty(0, T) \) \quad (by Lemma 2.1 and Theorem 4.1).

Since the strong solution \( u \) belongs to \( W^{2,1}_p(\Omega \times (0, T)) \), it follows from (4.14) that \( u \in L^p(\Omega \times (0, T)) \) with \( 1/p_1 = 1/2 - 2/(d+2) \); so that

(4.17) \( g(u) \in L^p(\Omega \times (0, T)). \)

We regard \( g(u) - \dot{C} \in L^p(\Omega \times (0, T)) \) (by (4.16) and (4.17)) as an inhomogeneous term in (4.15) and use the result in [2, pp. 341–342] to get

\[
v \in W^{2,1}_p(\Omega \times (0, T)) \quad \text{with} \quad p_s = \min \{ p, p_1 \}.
\]

If \( p_s < p \), then we repeat the above procedure; since \( u = v + C \in L^p(\Omega \times (0, T)) \) with \( p_s = 1/p_s - 2/(d+2) \) by (4.14) and, therefore, \( g(u) \in L^p(\Omega \times (0, T)) \), one can show \( v \in W^{2,1}_p(\Omega \times (0, T)) \) with \( p_s = \min \{ p, p_s \} \). After repeating this procedure finitely many
times, we arrive at the conclusion.

(ii) Since \( a \) belongs to \( W^2_p(\Omega) \) for every \( 2 \leq p < \infty \), it follows from (i) that \( u \in W^{2+1}_p(\Omega \times (0, T)) \). Therefore,

\[
u \in H^{\sigma, \theta}(\Omega \times [0, T]) \quad \text{for every } 0 < \sigma < 1
\]

(use [2, Lemma 3.3 in Chapter 2]); so that

\[
g(\nu) \in H^{\sigma, \theta}(\Omega \times [0, T]) \quad \text{for every } 0 < \sigma < 1.
\]

Moreover, \( t \to \dot{C}(t) \) is also Hölder continuous. Indeed, (2.2) and (2.3) imply that for every \( 0 < \theta < 1/2 \)

\[
|\dot{C}(t) - \dot{C}(s)| \leq K_\theta \left\| \frac{\partial u}{\partial t} (t) - \frac{\partial u}{\partial t} (s) \right\|^\theta \left\| \frac{\partial u}{\partial t} (t) - \frac{\partial u}{\partial t} (s) \right\|^{1-\theta}_{H^{\theta/2}}
\]

with some \( K_\theta > 0 \), from which one can deduce, with the aid of Theorem 4.1 and Remark 4.1,

\[
|\dot{C}(t) - \dot{C}(s)| \leq K''_\theta |t - s|^\theta
\]

for every \( 0 < \theta < 1/2 \),

with some \( K''_\theta > 0 \). The inhomogeneous term \( g(\nu) - \dot{\nu} \) in (4.15) being Hölder continuous in \( (x, t) \) by (4.18) and (4.19), it suffices to use the result of [2, Theorem 5.2 in Chapter 4] to complete the proof. q.e.d.

As an easy consequence of Proposition 4.6, we have

**Theorem 4.7.** If \( a \in C^\alpha(\Omega) \) and \( I \in C^\gamma[0, \infty) \) satisfy (4.1), \( a|_r = \text{constant} \) and \( \Delta a|_r = \text{constant} \), then (P) has a unique classical solution \( u \) such that \( \partial u/\partial t, \partial u/\partial \nu \text{ and } \Delta u \text{ are Hölder continuous in } (x, t) \).

§ 5. Semi-group approach.

We will briefly state that another approach based on analytic semi-group theory is also available for the study of (P).

In this section, the given functions \( a \) and \( I \) are assumed to satisfy, in place of (3.1) and (3.2),

\[
a \in L^\alpha(\Omega),
\]

\[
I \in C^{1+\theta}_r[0, \infty) \quad \text{with some } 0 < \theta \leq 1.
\]

We need some device to reduce (P) as the Cauchy problem for an abstract evolution equation in \( L^\alpha(\Omega) \). Take an auxiliary function \( \phi \), which is any smooth function on \( \Omega \) such that

\[
\phi|_r = \text{constant and } \int \frac{\partial \phi}{\partial \nu} d\Gamma = 1.
\]
If a new unknown function $v$ is defined by

$$
(5.3) \quad v = u - I(t)\phi,
$$

then the original problem (P) is reduced to

$$
(5.4) \quad \begin{cases}
\frac{\partial v}{\partial t} = \Delta v + h(t, v) & \text{in } \Omega \times (0, \infty) \\
v = \text{unknown function depending only on } t & \text{on } \Gamma \times (0, \infty), \\
\int_{\Gamma} \frac{\partial v}{\partial \nu} \, d\Gamma = 0 & \text{in } (0, \infty), \\
v(\cdot, 0) = b & \text{in } \Omega,
\end{cases}
$$

where $h(t, v) = I(t)\Delta \phi - I(t)\phi + g(v + I(t)\phi)$ and $b = a - I(0)\phi$. We also define a closed linear operator $A$ in $L^2(\Omega)$ by

$$
D(A) = \left\{ v \in H^2(\Omega); \, \forall v = \text{constant and } \int_{\Gamma} \frac{\partial v}{\partial \nu} \, d\Gamma = 0 \right\}
$$

$$
Av = -\Delta v \quad \text{for } v \in D(A).
$$

Since it is possible to show that $A$ is a non-negative self-adjoint operator in $L^2(\Omega)$, $-A$ generates an analytic semi-group of bounded linear operators $\{e^{-tA}\}_{t \geq 0}$. In terms of $A$, the reduced problem (5.4) is rewritten as

$$
(5.5) \quad \begin{align*}
\frac{dv}{dt} + Av &= h(t, v), & t &> 0, \\
v(0) &= b.
\end{align*}
$$

Here the Lipschitz continuity of $g$ and (5.2) imply that for every $T > 0$

$$
\| h(t, v) - h(s, w) \| \leq L_T (|t - s|^\alpha + \|v - w\|), \quad s, t \in [0, T], \quad v, w \in L^2(\Omega),
$$

with some $L_T > 0$.

Consider the mild form associated with (5.5):

$$
(5.6) \quad v(t) = e^{-tA}b + \int_0^t e^{-(t-s)A}h(s, v(s)) \, ds, \quad t \geq 0.
$$

From general results in the theory of evolution equations, (5.6) has a unique solution $v \in C([0, \infty); L^2(\Omega))$. Furthermore, such $v$ belongs to $C'((0, \infty); L^2(\Omega)) \cap C((0, \infty); D(A))$ and actually satisfies (5.5). For details, see, e.g., Henry [4].

Since $u$ is given by (5.3), we have

**Theorem 5.1.** There exists a unique function $u \in C([0, \infty); L^2(\Omega)) \cap C'((0, \infty); L^2(\Omega)) \cap C((0, \infty); H^2(\Omega))$, which satisfies (P).
Remark 5.1. If (5.1) is replaced by $a \in E \cap H'(\Omega)$ and the compatibility condition (4.1) holds, then the regularity of $u$ at $t=0$ is improved. In fact, the initial data $b$ of (5.5) being in $D(A)$, one can find that the solution $v$ of (5.6) is in $C^t([0, \infty); L^2(\Omega)) \cap C([0, \infty); H^s(\Omega))$, which, together with (5.3), implies that $u$ lies in the same class. This result is slightly better than the corresponding ones obtained in the preceding sections (cf. Theorem 3.1 and Lemma 4.3).

Remark 5.2. As is stated in Remark 5.1, the semi-group approach gives stronger results. However, the determination of unknown values of $u$ on $\Gamma$ is complicated because the free boundary condition is hidden in the abstract evolution equation (5.5). The authors believe that the semi-discretization method is not only elementary, but also useful from a practical point of view such as numerical analysis.

References


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