Bifurcation of Periodic Solutions for Nonlinear Parabolic Equations with Infinite Delays

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§ 1. Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \). We consider the following system of semilinear parabolic equations for \( u(x, t) = \text{col} (u^1(x, t), u^2(x, t), \ldots, u^n(x, t)) \) with \( x = (x_1, x_2, \ldots, x_n) \in \Omega \) and \( t \in \mathbb{R} \);

\[
\frac{\partial u}{\partial t}(x, t) = D(\alpha) \Delta u(x, t) + C(\alpha) u(x, t) + \int_{-\infty}^{t} K(t - \tau; \alpha) u(x, \tau) d\tau + f(u; \alpha)(x)
\]

in \( \Omega \times \mathbb{R} \),

where \( \alpha \) is a parameter moving on an open interval \( I \subseteq \mathbb{R} \), \( \Delta \) is the Laplace operator \( \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \), \( D(\alpha) \) is an \( N \times N \) diagonal matrix whose \( (i, i) \)-component \( d_i(\alpha) \) is positive for each \( i = 1, 2, \ldots, N \) and \( C(\alpha) = (c_{ij}(\alpha)), K(t; \alpha) = (k_{ij}(t; \alpha)) \) are \( N \times N \) real matrices. In (1.1), for each \( t \in \mathbb{R} \), \( u_t \) denotes an \( \mathbb{R}^n \)-valued function on \( \Omega \times (-\infty, 0] \) defined by \( u_t = u(x, t + \theta) \) with \( x \in \Omega \) and \( \theta \in (-\infty, 0] \) and, for every \( \alpha \in I \), the correspondence \( \phi \rightarrow f(\phi; \alpha) \) is a nonlinear mapping from \( C(\Omega \times (-\infty, 0]; \mathbb{R}^n) \) to \( C(\Omega \times \mathbb{R}^n) \) satisfying \( f(0; \alpha) = 0 \).

As boundary conditions for (1.1), we impose

\[
Bu = 0 \quad \text{on } \partial \Omega \times \mathbb{R},
\]

where \( Bu = u \) or \( Bu = \partial u / \partial \nu \) with \( \partial / \partial \nu \) the outward normal derivative on \( \partial \Omega \).

The purpose of the present paper is to construct non-trivial periodic solutions of (1.1)–(1.2) when \( \alpha \) varies over \( I \). As to Hopf bifurcation problems for ordinary functional differential equations (e.g., \( D(\alpha) \equiv 0 \) in (1.1)), there is a lot of existing literature. Especially, the monograph of Hale [6] contains various useful results on Hopf bifurcations and related problems for ordinary functional differential equations with finite delays. (See also the paper of Chafee [1], where careful analysis of bifurcation problems is carried out.) For the infinite delay case for ordinary functional differential equations, we refer the reader to the work of Simpson [14], [15], who has shown the existence of periodic solutions and discussed their stability. Recently, Yoshida and Kishimoto [19], [20] have studied Hopf bifurcation problems for partial
functional differential equations with finite time delays. They follow the idea of Hale [6] to get the existence of bifurcating solutions by using the center manifold theorem (see also Chow and Mallet-Paret [2]). However, it seems to the authors that there are only a few papers which discuss the existence of periodic solutions for partial functional differential equations with infinite delays (see Tesi [16]).

When we study the existence of non-trivial periodic solutions for (1.1)–(1.2), the periods are unknown. Therefore, we make it appear in the equations with use of a new parameter \( \omega > 0 \). By change of time scale,

\[
(1.3)
\]

\[ s = \omega t, \]

we will attempt to look for \( 2\pi \)-periodic (in \( s \)) solutions. By setting

\[
(1.4)
\]

\[ v(x, s) = u\left(x, \frac{s}{\omega}\right). \]

the original periodic problem is reduced to finding a non-trivial family \((v, \omega, \alpha)\) which satisfies

\[
(1.5)
\]

\[
\omega \frac{\partial v}{\partial s}(x, s) = D(\alpha)Dv(x, s) + C(\alpha)v(x, s)
\]

\[
+ \int_{-\infty}^{s} K(s - \sigma; \omega, \alpha)v(x, \sigma)d\sigma + f(v_{\omega}; \alpha)(x)
\]

in \( \Omega \times \mathbb{R} \),

\[
(1.6)
\]

\[ Bv = 0 \quad \text{on} \quad \partial \Omega \times \mathbb{R}, \]

\[
(1.7)
\]

\[ v(x, s) = v(x, s + 2\pi) \quad \text{in} \quad \Omega \times \mathbb{R}, \]

where

\[
(1.8)
\]

\[ K(s; \omega, \alpha) = \frac{1}{\omega} K\left(\frac{s}{\omega}; \alpha\right) \]

and

\[
(1.9)
\]

\[ v_{\omega}(x, \theta) = v(x, s + \omega \theta) \quad \text{for} \quad x \in \Omega, \quad \theta \in (-\infty, 0] \quad \text{and} \quad s \in \mathbb{R}. \]

Our procedure for constructing non-trivial solutions for (1.5)–(1.7) has some similarity to that of Sattinger [11], who discussed the Hopf bifurcation for the Navier-Stokes equations (see also Crandall and Rabinowitz [4]). We treat (1.5)–(1.7) in some function spaces of Schauder type, which consist of \( 2\pi \)-periodic functions, and use the implicit function theorem to get the existence assertions.

The content of this paper is as follows. In §2, we collect our hypotheses and state main theorems (Theorems I and II). In §3, we study a system of linear parabolic equations with Volterra integrals and give some preliminary results to develop
the bifurcation theory. § 4 is devoted to the proofs of Theorems I and II. In § 5 we investigate semilinear Volterra diffusion equations, which arise in population dynamics. Our bifurcation theory is applied to the study of these equations.

**Notation.**

Let $0<\gamma<1/2$. For every non-negative integer $k$, $C^{k+\gamma}(\bar{\Omega}; R^N)$ denotes the Banach space of all continuous functions $u: \bar{\Omega}\to R^N$ with continuous derivatives up to order $k$ and with finite norm

$$
\|u\|_{k+\gamma} = \sum_{|\beta|=k} \sum_{i=1}^N \sup_{x \in \bar{D}} |D^\beta u(x)| + \sum_{|\beta|=k} \sum_{i=1}^N \sup_{x, y \in \bar{D}} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^{\gamma}}.
$$

where $\beta=(\beta_1, \beta_2, \ldots, \beta_n)$ is a multi-index, $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$ and

$$
D^\beta = \left( \frac{\partial}{\partial x_1} \right)^{\beta_1} \left( \frac{\partial}{\partial x_2} \right)^{\beta_2} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\beta_n}.
$$

For $Q_T = \Omega \times (0, T)$, $C^{2k+2\gamma, k+\gamma}(Q_T; R^N)$ denotes the Banach space of all functions $u: Q_T \to R^N$ which are continuous together with all derivatives of the form $D^2 D^\beta u (i \geq 0$ is an integer and $D_i = \partial / \partial t_i$) for $|\beta| + 2j \leq 2k$ and have finite norm

$$
\|u\|_{2k+2\gamma, k+\gamma} = \sum_{0 \leq |\beta| + 2j \leq 2k} \sum_{i=1}^N \sup_{(x, \xi) \in Q_T} \frac{|D^\beta D^j u(x, t)|}{|x-y|^{\gamma}}
$$

and

$$
+ \sum_{|\beta| + 2j \leq 2k} \sum_{i=1}^N \sup_{(x, \xi) \in Q_T} \frac{|D^\beta D^j u(x, t) - D^\beta D^j u(y, t)|}{|x-y|^{\gamma}}
$$

(for more details, see, e.g., Ladyženskaja et al. [8]).

Furthermore, we introduce the following Banach spaces of $2\pi$-periodic functions. Let $H_2$ denote the Hilbert space of $2\pi$-periodic function $u: R \to \ell^2(\Omega; C^N)$ with inner product

$$
(u, v)_2 = \frac{1}{2\pi} \int_0^{2\pi} (u(s), v(s))_{L^2(\Omega; C^N)} ds
$$

$$
= \frac{1}{2\pi} \int_0^{2\pi} \langle u(x, s), v(x, s) \rangle dx ds,
$$

where $\langle \cdot, \cdot \rangle$ is defined by

$$
\langle a, b \rangle = \sum_{i=1}^N a_i b_i \quad \text{for } a, b \in C^N.
$$

The norm in $H_2$ is denoted by $\| \cdot \|_2$. For $k=0, 1, 2, \cdots$ and $0<\gamma<1/2$, $C^{2k+2\gamma, k+\gamma}_{2\pi}$
denotes the subspace of \( C^{2\pi+2\pi,\infty}(\mathbb{Q}; \mathbb{R})\), whose elements are \(2\pi\)-periodic in \(t\). The norm \(\|u\|_{2\pi+2\pi,\infty}\) in \(C_{2\pi}^{2\pi+2\pi,\infty}\) is given by

\[
\|u\|_{2\pi+2\pi,\infty} = \|u\|_{2\pi+2\pi,\infty}^{(2\pi)}
\]

for \(u \in C_{2\pi}^{2\pi+2\pi,\infty}\).

\section{Hypotheses and results}

First we collect some hypotheses on \(D(\alpha), C(\alpha)\) and \(K(t; \alpha)\).

(H.1) Every diagonal component of diagonal matrix \(D(\alpha)\) is positive and continuously differentiable in \(\alpha \in I\).

(H.2) \(C(\alpha)\) is a real \(N \times N\) matrix whose components are continuously differentiable in \(\alpha \in I\).

(H.3) \(K(t; \alpha)\) is a real \(N \times N\) matrix such that every component is continuously differentiable for \((t, \alpha) \in [0, \infty) \times I\). Moreover, there exists an integrable function \(\kappa\) over \([0, \infty)\) such that

\[
\max \{|K(t; \alpha)|, |D_t K(t; \alpha)|, |D_\alpha K(t; \alpha)|\} \leq \kappa(t)
\]

for all \(t \in [0, \infty)\) and \(\alpha \in I\). Here, for an \(N \times N\) matrix \(K\), \(|K|\) is given by

\[
|K| = \sup_{\theta + \omega \in \mathbb{R}^N} \frac{|K\theta|}{|\omega|}.
\]

To state hypotheses on \(f(\cdot; \alpha)\), it is convenient to put

\[
(2.1) \quad F(v, \omega, \alpha)(x, s) = f(v_{t, \omega}; \alpha)(x), \quad (x, s) \in \mathbb{Q} \times \mathbb{R},
\]

for \(v: \mathbb{Q} \times \mathbb{R} \to \mathbb{R}^N\) and \((\omega, \alpha) \in \mathbb{R}^* \times I\) with \(\mathbb{R}^* = (0, \infty)\), where \(v_{t, \omega}\) is defined by (1.9). Observe that, if \(v\) is \(2\pi\)-periodic in \(s\), then \(F(v, \omega, \alpha)\) is also \(2\pi\)-periodic in \(s\).

(H.4) (i) \(F\) is a \(C^1\)-mapping from \(C_{2\pi}^{2\pi+2\pi,1+\gamma} \times \mathbb{R}^* \times I\) to \(C_{2\pi}^{\gamma+\gamma}\) with some \(0 < \gamma < 1/2\) such that

\[
F(0, \omega, \alpha) = 0 \quad \text{and} \quad D_s F(0, \omega, \alpha) = 0,
\]

where \(D_s\) means the Fréchet derivative with respect to the \(v\)-argument.

(ii) For any bounded set \(V\) of \(C_{2\pi}^{2\pi+2\pi,1+\gamma} \times \mathbb{R}^* \times I\), \(\lim_{\epsilon \to 0} (1/\epsilon) D_s F(v, \omega, \alpha) = \lim_{\epsilon \to 0} (1/\epsilon) D_s F(v, \omega, \alpha) = 0\) uniformly for \((v, w, \alpha) \in V\). Moreover, \((1/\epsilon) D_s F(v, \omega, \alpha)\) converges uniformly for \((v, w, \alpha) \in V\) as \(\epsilon \to 0\).

For example, take a nonlinear mapping \(f\) of the form

\[
f(u, \alpha)(x) = g\left( u(x, t), u(x, t-r), \int_{-\infty}^{t} h(t-\tau) u(x, \tau) d\tau; \alpha \right),
\]

where \(r > 0\), \(h(t)\) and \(th(t) \in L^1(0, \infty)\) and \(g(u, v, w; \alpha)\) is a smooth function of \((u, v, w, \alpha)\) such that
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\[ g(0, 0, 0; \alpha) = D_u g(0, 0, 0; \alpha) = D_\omega g(0, 0, 0; \alpha) = D_{u_\omega} g(0, 0, 0; \alpha) = 0. \]

Since \( F \) is given by

\[ F(u, \omega, \alpha) = g\left(v, x, s; v(x, s - \omega r), \int_{-\infty}^{\infty} h(-\theta)v(x, s + w\theta)\,d\theta; \alpha\right), \]

it is easy to verify (H.4).

We now take the linear part of (1.1) and consider the corresponding "characteristic problem":

\[ \lambda w - D(\alpha) Dw - C(\alpha) w - \hat{K}(\lambda; \alpha) w = 0 \quad \text{in} \ \Omega, \]

\[ Bw = 0 \quad \text{on} \ \partial\Omega. \tag{2.2} \]

Here \( \hat{K}(\lambda; \alpha) \) is the Laplace transform of \( K(t; \alpha) \):

\[ \hat{K}(\lambda; \alpha) = \int_0^{\infty} e^{-\lambda t} K(t; \alpha) \, dt, \]

which is analytic in \( \lambda \) for \( \text{Re} \lambda > 0 \) and continuously differentiable in \( \lambda \) for \( \text{Re} \lambda \geq 0 \) (use (H.3)). It is known that \( u \equiv 0 \) is asymptotically stable for (1.1)-(1.2) in some suitable function spaces if the characteristic problem (2.2) has no characteristic values \( \lambda \) with \( \text{Re} \lambda \geq 0 \). (We say that \( \lambda \) is a characteristic value of (2.2) if (2.2) has non-trivial solutions \( w \) for such \( \lambda \).) For details, see Schiaffino and Tesei [13] and Yamada [18].

The above stability result implies that no nontrivial periodic solutions of (1.1)-(1.2) appear in a neighbourhood of \( u \equiv 0 \) when (2.2) has no characteristic values \( \lambda \) in the plane \( \text{Re} \lambda \geq 0 \). Therefore, we will treat the case where a characteristic value \( \lambda \) of (2.2) lies on the imaginary axis. More precisely, let \( \{\mu_m\}_{m=0}^{\infty} \) with \( \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \) be eigenvalues of

\[ -\Delta u = \mu_0 u \quad \text{in} \ \Omega \quad \text{and} \quad Bu = 0 \quad \text{on} \ \partial\Omega, \tag{2.3} \]

where \( u \) is scalar-valued. Let \( \phi_m \) denote the eigenfunction associated with \( \mu_m \). We may take \( \{\phi_m\}_{m=0}^{\infty} \) as a complete orthonormal system in \( L_2(\Omega; \mathbb{R}) \). Substitution of \( w = \sum_{m=0}^{\infty} \phi_m a_m \) with \( a_m \in C^\infty \) into (2.2) leads to the following equivalent problem

\[ E_m(\lambda, \alpha) a_m = 0 \quad \text{for} \ m = 0, 1, 2, \cdots, \]

where

\[ E_m(\lambda, \alpha) = \lambda I + \mu_m D(\alpha) - C(\alpha) - \hat{K}(\lambda; \alpha), \tag{2.4} \]

with \( I \) the identity matrix.

We assume

(H.5) For some \( \alpha_0 \in I \), there exist a positive number \( \omega_0 \) and a positive integer \( M \) such that

\[ E_m(\lambda, \alpha_0) \neq 0 \quad \text{for} \ m = 0, 1, 2, \cdots, \]

\( m \leq M \).
rank \(E_M(i\omega_0, \alpha_0) = N - 1\)

and

\[
\det E_m(i l\omega_0, \alpha_0) \neq 0 \quad \text{for all} \quad (l, m) \neq (1, M)
\]

with \(l = 0, 1, 2, \ldots\) and \(m = 0, 1, 2, \ldots\).

It follows from (H.5) that

\[\dim \text{ Ker } E_M(i\omega_0, \alpha_0) = 1 \quad \text{and} \quad \dim \text{ Ker } E_M(i\omega_0, \alpha_0)^* = 1,\]

where \(\text{ Ker } E\) denotes the kernel of \(E\) and \(E^*\) denotes the adjoint matrix of \(E\).

Furthermore, we assume

(\text{H.6}) For some non-zero vectors \(a_M \in \text{ Ker } E_M(i\omega_0, \alpha_0)\) and \(a_M^* \in \text{ Ker } E_M(i\omega_0, \alpha_0)^*\),

\[
\left\langle D_M E_M(i\omega_0, \alpha_0) a_M, a_M^* \right\rangle \neq 0,
\]

where \(D_M E_M(i\omega_0, \alpha_0) = \text{Id} - D_M \hat{K}(i\omega_0; \alpha_0)\).

(\text{H.7}) For some non-zero vectors \(a_M \in \text{ Ker } E_M(i\omega_0, \alpha_0)\) and \(a_M^* \in \text{ Ker } E_M(i\omega_0, \alpha_0)^*\),

\[
\text{Re}\left\{ \left\langle D_M E_M(i\omega_0, \alpha_0) a_M, a_M^* \right\rangle \right\} \neq 0,
\]

where \(D_M E_M(i\omega_0, \alpha_0) = \mu_M D_M D(\alpha_0) - D_M C(\alpha_0) - D_M \hat{K}(i\omega_0; \alpha_0)\).

We are ready to state our main results.

\textbf{Theorem I.} Assume (H.1)–(H.7). Then for some sufficiently small \(\varepsilon_0 > 0\), there exists a family of functions \((u(\varepsilon)(x, s), \omega(\varepsilon), \alpha(\varepsilon)) \in C^1([[-\varepsilon_0, \varepsilon_0]; C^2 \times \mathbb{R} \times \mathbb{R})\) such that \(\omega(0) = \omega_0, \alpha(0) = \alpha_0\) and

\[u(x, \tau; \varepsilon) = \varepsilon u(\varepsilon)(x, \omega(\varepsilon) \tau)\]

is a \(2\pi \omega(\varepsilon)^{-1}\)-periodic solution of (1.1)–(1.2) with \(\alpha = \alpha(\varepsilon)\).

\textbf{Remark 2.1.} Our hypothesis (H.5) does not necessarily imply that \(\text{ Ker } E_M(i\omega_0, \alpha_0)\) coincides with \(\text{ Gen Ker } E_M(i\omega_0, \alpha)\), where for an \(N \times N\) matrix \(E\) \(\text{ Gen Ker } E\) means the generalized kernel of \(E\) defined by

\[\text{ Gen Ker } E = \bigoplus_{j=1}^{\infty} \text{ Ker } E^j.\]

(Clearly, \(\text{ Gen Ker } E = \text{ Ker } E^j\) for some \(j\)) It is easy to see that

\[
\left\langle a_M, a_M^* \right\rangle \neq 0 \quad \text{for} \quad 0 \neq a_M \in \text{ Ker } E_M(i\omega_0, \alpha_0) \quad \text{and} \quad 0 \neq a_M^* \in \text{ Ker } E_M(i\omega_0, \alpha_0)^*
\]

if \(\dim \text{ Gen Ker } E_M(i\omega_0, i\alpha_0) = 1\), while
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\[ \langle a, a^*_n \rangle = 0 \quad \text{for } 0 \neq a \in \text{Ker } E(i\omega, \alpha) \quad \text{and} \]
\[ 0 \neq a^*_n \in \text{Ker } E(i\omega, \alpha)^* \]

if \( \text{dim Gen ker } E(i\omega, \alpha) > 1 \). Therefore, in the case where \( K(t; \alpha) \equiv 0 \), (H.6) is equivalent to

\[ \text{dim Gen ker } E(i\omega, \alpha) = 1, \]

which is one of the conditions assumed in the standard Hopf bifurcation theory (see Crandall and Rabinowitz [4]).

**Remark 2.2.** By virtue of (H.6), it will be proved that there exists a unique function \( \lambda(\alpha) \in C^1(I'; C) \) with some open interval \( I' (\exists \alpha_0) \) in \( I \) such that

\[ \det E(i\alpha) = 0 \quad \text{and} \quad \lambda(\alpha) = i\omega. \]

(See Lemma 4.1.)

In place of (H.7), we can also give another useful hypothesis which is expressed in terms of \( \lambda(\alpha) \) given in Remark 2.2.

**Theorem II.** In addition to (H.1)–(H.6), assume the following condition;

(H.8) \( \) The function \( \lambda(\alpha) \in C^1(I'; C) \) with \( \alpha_0 \in I' \subset I \) satisfying (2.5) also satisfies

\[ \text{Re } D_\alpha \lambda(\alpha_0) \neq 0. \]

Under these conditions, the same conclusion as in Theorem I holds true.

**Remark 2.3.** We can weaken (H.4) slightly. Indeed, even if the convergence of \( e^{-1}D_\varepsilon F(\varepsilon t, \omega, \alpha) \) as \( \varepsilon \to 0 \) is not assumed, the conclusions of Theorems I and II remain true with the \( C^1 \)-dependence of \( (t(\varepsilon), \omega(\varepsilon), \alpha(\varepsilon)) \) with respect to \( \varepsilon \) replaced by the continuous dependence. See Remark 4.1.

**Remark 2.4.** In this paper, Theorems I and II will be proved by the implicit function theorem in §4. However, if we use some topological properties of the linear part of (1.5)–(1.7), we can weaken (H.4)–(H.7) to show the existence of periodic solutions for the case when a characteristic value \( \lambda = i\omega_0 \) of (2.2) at \( \alpha = \alpha_0 \) has multiplicity greater than (or equal to) one. Indeed, a sufficient condition for Hopf bifurcation is expressed in terms of the topological degree:

\[ \sum_{i=1}^m \sum_{n=0}^\infty \text{deg } (\eta_{i,m}, U, 0) \neq 0, \]

where \( \eta_{i,m} \) is a mapping from \( \mathbb{R}^2 \) to \( C \) (identified with \( \mathbb{R}^2 \)) given by \( (\omega, \alpha) \to \det E(i\omega, \alpha) \) and \( U \) is a neighbourhood of \( (\omega_0, \alpha_0) \) such that \( \eta_{i,m}(\partial U) \neq 0 \). These results will be published elsewhere [10].
§ 3. Study of linearized problem

In this section we will discuss the following linearized problem associated with (1.5)–(1.7) for \( \omega = \omega_0 \) and \( \alpha = \alpha_0 \):

\[
(3.1) \quad \omega_0 \frac{\partial v}{\partial s} = D(\alpha_0) \Delta v + C(\alpha_0)v + \int_{-\infty}^{\tau} K(s-\sigma; \omega_0, \alpha_0)v(\sigma)d\sigma + g \quad \text{in } \Omega \times \mathbb{R},
\]

\[
(3.2) \quad Bv = 0 \quad \text{on } \partial \Omega \times \mathbb{R},
\]

\[
(3.3) \quad v(\cdot, s) = v(\cdot, s+2\pi) \quad \text{in } \Omega \times \mathbb{R}.
\]

In view of (3.3), we introduce two Banach spaces

\[
X = \{ u \in C^{2+\gamma, 1+\gamma}_{2\pi}; \ Bv = 0 \text{ on } \partial \Omega \times \mathbb{R} \} \quad \text{and} \quad Y = C^{\gamma, \gamma}_{2\pi}
\]

with \( 0 < \gamma < 1/2 \).

Our first result is concerned with the solvability of

\[
(3.4) \quad \omega_0 \frac{\partial v}{\partial s} - D(\alpha_0) \Delta v - C(\alpha_0)v + \zeta v = g \quad \text{in } \Omega \times \mathbb{R}
\]

in the space \( X \).

**Lemma 3.1.** Let \( g \) be in \( Y \). If \( \zeta > 0 \) is sufficiently large, then there exists a unique \( v \in X \) satisfying (3.4). If the transformation: \( g \to v \) is denoted by \( R_0(\zeta) \), then

\[
\| R_0(\zeta)g \|_{2^\gamma + 1, 1+\gamma} \leq C(\zeta)\| g \|_{2^\gamma, \gamma}
\]

with some constant \( C(\zeta) \) depending on \( \zeta \).

**Proof.** We will prove this lemma by taking \( \omega_0 = 1 \) without loss of generality. Let \( S_0(s; \zeta) (s \geq 0) \) denote the transformation: \( v_0 \to v(s; v_0) \), where \( v(s; v_0) \) is the unique solution of (3.4) with \( g \equiv 0 \), (3.2) and \( v(0; v_0) = v_0 \). In terms of \( S_0(s; \zeta) \), the general solution \( v \) of (3.4) and (3.2) can be represented as

\[
(3.5) \quad v(s) = S_0(s; \zeta)v(0) + \int_{0}^{s} S_0(s-\sigma; \zeta)g(\sigma)d\sigma.
\]

Moreover, the à priori estimate of Schauder type yields

\[
(3.6) \quad \| v \|_{2^\gamma + 1, 1+\gamma} \leq C(\zeta, T)(\| v(0) \|_{2^\gamma + 1} + \| g \|_{2^\gamma, \gamma})
\]

for any \( T > 0 \) with some constant \( C(\zeta, T) \) (see, e.g., Ladyženskaya et al. [8]). In view of (3.5), the condition that \( v \) is \( 2\pi \)-periodic leads to

\[
(3.7) \quad (1 - S_0(2\pi; \zeta))v(0) = \int_{0}^{2\pi} S_0(2\pi-\sigma; \zeta)g(\sigma)d\sigma.
\]
Now we note \( S_0(2\pi; \zeta) = e^{-\pi \zeta} S_0(2\pi; 0) \); so that the operator norm of \( S_0(2\pi; \zeta) \) in \( C^{\infty}(\bar{\Omega}; R^N) \) becomes less than 1 if \( \zeta \) is sufficiently large. Hence it follows that \((1 - S_0(2\pi, \zeta))\) has a bounded inverse operator in \( C^{\infty}(\bar{\Omega}; R^N) \) for every sufficiently large \( \zeta \). Moreover, \( v(0) \) is uniquely determined from (3.7) and it satisfies

\[
\|v(0)\|_{2\pi} \leq C_1(\zeta) \left\| \int_0^{2\pi} S_0(2\pi - \sigma; \zeta) g(\sigma) d\sigma \right\|_{2\pi} \\
\leq C_1(\zeta) C(\zeta, 2\pi) \|g\|_{2\pi, r}
\]

with some \( C_1(\zeta) > 0 \), where (3.6) with \( v(0) \equiv 0 \) has been used.

Here we invoke the smoothing effects for parabolic equations; in particular,

\[
\|S_0(2\pi; \zeta) w\|_{2\pi, r} \leq C_2(\zeta) \|w\|_{2\pi} \quad \text{for } w \in C^{\infty}(\bar{\Omega}; R^N)
\]

(see, e.g., Henry [7] or Sattinger [11, (5.2.5)]). By making use of (3.7) again, it follows from (3.6), (3.8) and (3.9) that

\[
\|v(0)\|_{2\pi} \leq \left\| S_0(2\pi, \zeta) v(0) \right\|_{2\pi} + \left\| \int_0^{2\pi} S_0(2\pi - \sigma; \zeta) g(\sigma) d\sigma \right\|_{2\pi, r} \\
\leq C_2(\zeta) \|v(0)\|_{2\pi} + C(\zeta, 2\pi) \|g\|_{2\pi, r} \\
\leq C(\zeta, 2\pi)(C_1(\zeta) C_2(\zeta) + 1) \|g\|_{2\pi, r}.
\]

The estimate in Lemma 3.1 is derived from (3.6) (with \( T = 2\pi \)) and (3.10). \( \text{q.e.d.} \)

If we define a bounded linear operator \( J_\zeta: X \to Y \) by

\[
J_\zeta v = \omega_\zeta \frac{\partial v}{\partial s} - D(\alpha_\zeta) \Delta v - C(\alpha_\zeta) v \quad \text{for } v \in X,
\]

then Lemma 3.1 implies that for sufficiently large \( \zeta \), \( J_\zeta + \zeta \) has a bounded inverse \( R_\zeta(\zeta) = (J_\zeta + \zeta)^{-1} \), which is compact in \( Y \).

**Lemma 3.2.** If \( K_\zeta \) is defined by

\[
(K_\zeta v)(x, s) = \int_{-\infty}^{s} K(s - \sigma; \omega_\zeta, \alpha_\zeta) v(x, \sigma) d\sigma,
\]

where \( K(s; \omega_\zeta, \alpha_\zeta) \) is given by (1.8), then \( K_\zeta \) is a bounded linear operator from \( X \) (resp. \( Y \)) to \( X \) (resp. \( Y \)).

**Proof.** Since

\[
(K_\zeta v)(x, s) = \int_{0}^{s} K(\theta; \omega_\zeta, \alpha_\zeta) v(x, s - \theta) d\theta,
\]

we see that
\[ D_x K_0^2 v = K_0^2 D_x v \quad \text{and} \quad D_x K_0 v = K_0^2 D_x v. \]

Therefore, the assertions easily follow from (H.3) and the definitions of \( X \) and \( Y \).

q.e.d.

We set

\[ L_0 = J_0 - K_0, \]

where \( J_0 \) and \( K_0 \) are defined by (3.11) and (3.12), respectively. By Lemma 3.2, \( L_0 \) is a bounded linear operator from \( X \) to \( Y \). We will determine the null space and the range space of \( L_0 \), which are denoted by \( N(L_0) \) and \( R(L_0) \), respectively.

For this purpose, it is convenient to introduce the following operator \( \tilde{L}_0 \) in the complex Hilbert space \( H_{2x} \):

\[
D(\tilde{L}_0) = \{ v \in H_{2x}; v \in L^1_{\text{loc}}(R; H^1(\Omega; C^N)), \partial v/\partial s \in L^1_{\text{loc}}(R; L^2(\Omega; C^N)) \quad \text{and} \quad Bv = 0 \ \text{a.e. on} \ \partial \Omega \times R \},
\]

\[
(\tilde{L}_0 v)(s) = \omega_0 \frac{\partial v}{\partial s}(s) - D(\alpha_0)\Delta v(s) - C(\alpha_0)v(s) - \int_{-\infty}^{s} K(s - \sigma; \omega_0, \alpha_0) v(\sigma) d\sigma
\]

for \( v \in D(\tilde{L}_0) \).

By Proposition A in Appendix, \( \tilde{L}_0 \) is a closed linear operator with dense domain. The adjoint operator of \( \tilde{L}_0 \) is denoted by \( \tilde{L}_0^* \). Since \( \tilde{L}_0 u = L_0 u \) for \( u \in X \), \( \tilde{L}_0 \) will play an auxiliary role in the study of \( L_0 \).

We have

\textbf{Lemma 3.3.} Let \( a_M^\phi \) and \( a_M^\psi \) be any non-trivial vectors of \( \text{Ker} \ E_{\psi}(i\omega_0, \alpha_0) \) and \( \text{Ker} \ E_{\psi}(i\omega_0, \alpha_0)^* \), respectively. Then

\[ N(\tilde{L}_0) = \{ e^{it\phi_M a_M^\phi}, e^{-it\psi_M a_M^\psi} \} \]

and

\[ N(\tilde{L}_0^*) = \{ e^{it\phi_M a_M^\phi}, e^{-it\psi_M a_M^\psi} \}, \]

where \( a \) denotes the complex conjugate of \( a \in C^N \).

\textbf{Proof.} Let \( u \in N(\tilde{L}_0) \). Since \( u \) is expressed as

\[
u = \sum_{l=-\infty}^{\infty} \sum_{m=0}^{\infty} e^{it\phi_M a_M} \quad \text{with} \quad a_M \in C^N,
\]

operating \( \tilde{L}_0 \) to (3.14) yields

\[
\tilde{L}_0 u = \sum_{l=-\infty}^{\infty} \sum_{m=0}^{\infty} e^{it\phi_M} E_M(il\omega_0, \alpha_0) a_M = 0
\]
(recall the definition (2.4)). Therefore,

\[(3.15) \quad E_m(il\omega_0, \alpha_0) \gamma_{lm} = 0 \quad \text{for } l = 0, \pm 1, \pm 2, \ldots \quad \text{and } m = 0, 1, 2, \ldots.\]

Since \( E_m(il\omega_0, \alpha_0) = E_m(-il\omega_0, \alpha_0) \), it follows from (H.5) and (3.15) that \( u \) is written as a linear combination of \( e^{i\tau} \phi_{y} a_{M} \) and \( e^{-i\tau} \phi_{y} a_{M} \) with \( a_{M} \in \text{Ker} \ E_{\tau}(i\omega_0, \alpha_0) \). These two terms are orthogonal in \( H_{x} \); so that we get the conclusion.

In order to determine \( N(\tilde{L}_{y}) \), it is sufficient to repeat the above procedure with use of Proposition A. q.e.d.

Let \( H^R_{x} \) denote the space of all functions \( v \in H_{x} \) such that \( v \) are \( R^\mathbb{C} \)-valued. Let \( X' \) and \( Y' \) be the dual spaces of \( X \) and \( Y \), respectively. Since both \( X \) and \( Y \) are densely embedded in \( H^R_{x} \), one can obtain the following inclusion relations;

\[ H^R_{x} \subset X' \quad \text{and} \quad H^R_{x} \subset Y', \]

where each inclusion is continuous. We define the injection \( I_{x} : H^R_{x} \rightarrow X' \) by

\[(3.16) \quad I_{x} g(u) = (g, u)_{x} \quad \text{for } u \in X \quad \text{and} \quad g \in H^R_{x} \]

and the injection \( I_{y} : H^R_{y} \rightarrow Y' \) by

\[(3.17) \quad I_{y} g(u) = (g, u)_{y} \quad \text{for } u \in Y \quad \text{and} \quad g \in H^R_{y}. \]

Let \( L_{0}' : Y' \rightarrow X' \) be the dual operator of \( L_{0} \). Then we have

**Lemma 3.4.** If \( v \in D(\tilde{L}_{0}) \cap H^R_{x} \), then

\[ L_{0}' I_{x} v = I_{x} \tilde{L}_{0} v. \]

**Proof.** We first note \( D(\tilde{L}_{0}) = D(\bar{L}_{0}) \) by Proposition A. Then it is sufficient to observe that the following identities hold for all \( u \in X' ; \)

\[
(L_{0}' I_{y} v)(u) = I_{y} v(L_{0} u) = (L_{0} u, v)_{x} = (\tilde{L}_{0} u, v)_{x} = (u, \tilde{L}_{0} v)_{x} = (I_{y} \tilde{L}_{0} v)(u),
\]

where (3.16) and (3.17) have been used. q.e.d.

In what follows, we fix any specific non-trivial vectors \( a_{M} \in \text{Ker} \ E_{\tau}(i\omega_0, \alpha_0) \) and \( a_{M}^{*} \in \text{Ker} \ E_{\tau}(i\omega_0, \alpha_0)^{*} \).

**Lemma 3.5.** (i) \( N(L_{0}) = \{ \psi_{1}, \psi_{2} \} \), where

\[
\psi_{1} = e^{i\tau} \phi_{y} a_{M} + e^{-i\tau} \phi_{y} a_{M}^{*} \quad \text{and} \quad \psi_{2} = \frac{1}{i} [e^{i\tau} \phi_{y} a_{M}^{*} - e^{-i\tau} \phi_{y} a_{M}].
\]
(ii) \( N(L_0^\ast) = \{ I_Y \cdot \psi_1^\ast, I_Y \cdot \psi_2^\ast \} \), where

\[
\psi_1^\ast = e^{i \zeta} \phi_M a_M^\ast + e^{-i \zeta} \phi_M a_M^\ast \quad \text{and} \quad \psi_2^\ast = \frac{1}{i} \{ e^{i \zeta} \phi_M a_M^\ast - e^{-i \zeta} \phi_M a_M^\ast \}.
\]

**Proof.** Let \( u \in N(L_0) \). Since \( \tilde{L}_0 \) is an extension of \( L_0 \), we have \( u \in N(\tilde{L}_0) \cap H_2^\infty \). By Lemma 3.3, \( u \) can be expressed in the form

\[
u = \beta e^{i \zeta} \phi_M a_M + \beta^{-1} e^{-i \zeta} \phi_M a_M \quad \text{with} \quad \beta \in C.
\]
(Note that \( u \) is \( R^X \)-valued.) This fact implies that \( u \) is written as a linear combination of \( \psi_1 \) and \( \psi_2 \) over \( R \). Furthermore, both \( \psi_1 \) and \( \psi_2 \) belong to \( X \) because it is well known that \( \phi_M \) is smooth on \( \partial \). Thus the proof of (i) is complete.

We use the following identities

\[
L_0 = (J_0 + \zeta) - (K_0 + \zeta)
= [1 - (K_0 + \zeta) R_0(\zeta)](J_0 + \zeta)
\tag{3.18}
\]
with sufficiently large \( \zeta \),

to show (ii). Here \( J_0 \) is defined by (3.11) and \( R_0(\zeta) = (J_0 + \zeta)^{-1} \). By virtue of Lemma 3.1, it is easy to see that

\[
\dim N(L_0) = \dim N(1 - [(K_0 + \zeta) R_0(\zeta)'])
\]
(Carry out the dual operation for (3.18).) By Lemmas 3.1 and 3.2, \( (K_0 + \zeta) R_0(\zeta) \) is a compact operator in \( Y \); so that the Riesz-Schauder theory (see, e.g., Yosida [21, Theorem 3 in Chapter 10]), together with (i) of this lemma, implies

\[
\dim N(L_0^\ast) = \dim N(L_0) = 2.
\tag{3.19}
\]
Since \( \psi_1^\ast \) and \( \psi_2^\ast \) are linearly independent vectors in \( N(\tilde{L}_0^\ast) \cap H_2^\infty \) by Lemma 3.3, one can easily arrive at the conclusion with use of Lemma 3.4 and (3.19). q.e.d.

**Lemma 3.6.** \( R(L_0) = \{ u \in Y; (u, \psi_j^\ast)_{2\infty} = 0 \text{ for } j = 1, 2 \} \).

**Proof.** We first recall the expression (3.18). For every sufficiently large \( \zeta \), \( (K_0 + \zeta) R_0(\zeta) \) is compact in \( Y \) and \( (J_0 + \zeta) \) is an isomorphism from \( X \) to \( Y \); so that the Riesz-Schauder theory yields the closedness of \( R(L_0) \) ([21, p. 283]). We next apply the closed range theorem (see [21, p. 205]) to get

\[
R(L_0) = N(L_0^\ast)^{-1} \equiv \{ u \in Y; g(u) = 0 \text{ for all } g \in N(L_0^\ast) \}
= \{ u \in Y; I_Y \cdot \psi_j^\ast(u) = 0 \text{ for } j = 1, 2 \}
\]
where Lemma 3.5(ii) has been used. To accomplish the proof, it is sufficient to note

\[
I_Y \cdot \psi_j^\ast(u) = (\psi_j^\ast, u)_{2\infty} = (u, \psi_j^\ast)_{2\infty}, \quad j = 1, 2.
\]
q.e.d.
We are now able to decompose $X$ and $Y$ in a suitable manner. The Hahn-Banach theorem assures the existence of $g_j \in X'(j=1, 2)$ such that

$$g_j(\psi_k) = \delta_{jk} \text{ (Kronecker's delta)} \quad \text{for } j, k = 1, 2.$$  

(3.20)

For each $u \in X$, define

$$Pu = g_1(u)\psi_1 + g_2(u)\psi_2.$$  

(3.21)

It is easy to see that $P$ is a projection from $X$ onto $N(L_0)$. Thus we can obtain the following decomposition,

$$X = X_0 \oplus X_1 \quad \text{with} \quad X_0 = PX \quad \text{and} \quad X_1 = (1 - P)X;$$  

(3.22)

that is, any $u \in X$ is uniquely expressed in the form $u = u_0 + u_1$ with $u_0 \in X_0$ and $u_1 \in X_1$. Since $\text{codim } R(L_0) = 2$ by Lemma 3.6, it is also possible to choose $u_j \in Y (j=1, 2)$ such that

$$\langle u_j, \psi_k^* \rangle_{X_0} = \delta_{jk} \quad \text{for } j, k = 1, 2.$$  

(3.23)

For each $u \in Y$, we define the following projection $Q$ in $Y$:

$$Qu = \langle u, \psi_1^* \rangle_{X_0} u_1 + \langle u, \psi_2^* \rangle_{X_0} u_2.$$  

(3.24)

If $Y$ is decomposed as

$$Y = Y_0 \oplus Y_1 \quad \text{with} \quad Y_0 = QY \quad \text{and} \quad Y_1 = (1 - Q)Y,$$

(3.25)

then $Y_1$ is identical with $R(L_0)$.

**Lemma 3.7.** If the restriction of $L_0$ on $X_1$ is denoted by $L_1$, then $L_1$ has a bounded inverse from $Y_1$ to $X_1$.

**Proof.** Since the bounded linear operator $L_1$ gives a one-to-one mapping from $X_1$ onto $Y_1$, the conclusion follows from the open mapping theorem (see Yosida [21, p. 75–77]). q.e.d.

**Remark 3.1.** By a simple calculation, one can show

$$\langle \psi_1, \psi_2^* \rangle_{X_0} = \langle \psi_2, \psi_1^* \rangle_{X_0} = 2 \text{ Re } \langle a_M, a_M^* \rangle$$  

(3.26)

and

$$\langle \psi_1, \psi_2^* \rangle_{X_0} = -\langle \psi_2, \psi_1^* \rangle_{X_0} = -2 \text{ Im } \langle a_M, a_M^* \rangle.$$  

If $\dim \text{ Gen ker } E_M(i\omega_0, \alpha_0) = 1$, then we may take $\langle a_M, a_M^* \rangle = 1/2$ (see Remark 2.1). Therefore, (3.26) enables us to choose $I_{-1} \psi_f^* = g_j$ satisfying (3.20) and choose $\psi_j$ as $u_j$ satisfying (3.23). In this case, both $P$ and $Q$ (defined by (3.21) and (3.24)) have the same form.
\( (\cdot, \psi_3)_{L^2} \phi_1 + (\cdot, \phi_3^\parallel)_{L^2} \phi_2. \)

In view of Lemma 3.6, our decomposition (3.22) is furnished with \( X_0 = N(L_0) \) and \( X_1 = R(L_0) \cap X \). Analogously, \( Y_0 = N(L_0) \) and \( Y_1 = R(L_0) \) in decomposition (3.25).

\section{Proofs of Theorems}

\subsection{Proof of Theorem I}

We will consider (1.5)–(1.6) in place of (1.1)–(1.2) and construct non-trivial 2\( \pi \)-periodic solutions. With use of a new parameter \( \varepsilon \), we set

\begin{equation}
\begin{aligned}
&v = \varepsilon v_1
\end{aligned}
\end{equation}

and rewrite (1.5) in the form

\begin{equation}
\begin{aligned}
&L(\omega, \alpha)v_1(s) + G(v_1, \omega, \alpha; \varepsilon)(s) = 0.
\end{aligned}
\end{equation}

In (4.2), \( L(\omega, \alpha) \) is a bounded linear operator from \( X \) to \( Y \) defined by

\begin{equation}
\begin{aligned}
&L(\omega, \alpha)v(s) = \omega \frac{\partial v}{\partial s}(s) - D(\alpha)Jv(s) - C(\alpha)v(s) - \int_{-\infty}^{s} K(s - \sigma; \omega, \alpha)v(\sigma)d\sigma
\end{aligned}
\end{equation}

and \( v \rightarrow G(v, \omega, \alpha; \varepsilon) \) is a nonlinear operator from \( X \) to \( Y \) defined by

\begin{equation}
\begin{aligned}
&G(v, \omega, \alpha; \varepsilon) = \frac{1}{\varepsilon} F(\varepsilon v, \omega, \alpha),
\end{aligned}
\end{equation}

where \( F \) is given by (2.1). By virtue of (H.4), \( G \) is a \( C^1 \)-mapping from \( X \times \mathbb{R}^* \times I \times \mathbb{R} \) to \( Y \) satisfying

\begin{equation}
\begin{aligned}
&G(v, \omega, \alpha; 0) = D_v G(v, \omega, \alpha; 0) = D_\omega G(v, \omega, \alpha; 0) = D_\alpha G(v, \omega, \alpha; 0) = 0.
\end{aligned}
\end{equation}

We attempt to construct a one-parameter family \( \{(v_1(\varepsilon), \omega(\varepsilon), \alpha(\varepsilon)); \varepsilon \in \mathbb{R}\} \), which satisfies (4.2) for each \( \varepsilon \) with \( v_1(\varepsilon) \) lying in the space \( X \). Our strategy is to make use of decompositions (3.22) and (3.25) to apply the implicit function theorem. Since any element \( v_0 \) of \( X_0 \) can be expressed as \( v_0 = p\psi_1 + q\psi_2 \) with \( p, q \in \mathbb{R} \), one can also write, by recalling the definition of \( \psi_\varepsilon \) (see Lemma 3.5),

\begin{equation}
\begin{aligned}
&v_0(s) = 2\rho \Re \{e^{i(s+\delta)}\phi_\omega a_y\} = \rho \psi_1(s + \delta),
\end{aligned}
\end{equation}

with \( pe^{i\delta} = p - iq \). Equation (4.2) has the translation invariance with respect to \( s \)-variable; so that, after an appropriate phase shift, we may look for \( v_1 \) in the form

\begin{equation}
\begin{aligned}
&v_1 = \psi_1 + w \quad \text{with} \quad w \in X_1.
\end{aligned}
\end{equation}

(We understand that the parameter \( \varepsilon \) in (4.1) represents the amplitude of the oscillations.)
Substitution of (4.4) into (4.2) leads to

\[(4.5) \quad L_0 w + (L(\omega, \alpha) - L(\omega_0, \alpha_0))(\psi_1 + w) + G(\psi_1 + w, \omega, \alpha; \varepsilon) = 0.\]

Operating \(Q\) and \(1 - Q\) to (4.5) and using Lemmas 3.6 and 3.7, we get the following equations equivalent to (4.5):

\[(4.6) \quad (L(\omega, \alpha) - L(\omega_0, \alpha_0))(\psi_1 + w) + G(\psi_1 + w, \omega, \alpha; \varepsilon), \psi^*)_{2\pi} = 0 \quad \text{with} \quad \psi^*(s) = \psi^*_1(s) + i\psi^*_2(s) = 2e^{i\alpha} \phi_\mu a_\mu^* ,
\]

and

\[(4.7) \quad w + L^{-1}_1(1 - Q)[(L(\omega, \alpha) - L(\omega_0, \alpha_0))(\psi_1 + w) + G(\psi_1 + w, \omega, \alpha; \varepsilon)] = 0.\]

Here we observe that the following identities hold for \(v \in X\):

\[\left(\frac{\partial v}{\partial s}, \psi^*\right)_{2\pi} = i(v, \psi^*)_{2\pi},\]

\[(D(\alpha)Dv, \psi^*)_{2\pi} = - (\mu M D(\alpha)v, \psi^*)_{2\pi},\]

\[\left(\int_{-\pi}^{\pi} K(s - \sigma; \omega, \alpha) v(\sigma)d\sigma, \psi^*\right)_{2\pi} = (\hat{K}(i\omega; \alpha)v, \psi^*)_{2\pi}.\]

(For the derivation of the last identity, it is sufficient to use (1.8) and change the order of integration.) Therefore, (4.6) can be rewritten equivalently in the form

\[(4.8) \quad i(\omega - \omega_0)(\psi_1 + w, \psi^*)_{2\pi} + ([\mu M(D(\alpha) - D(\alpha_0)) - (C(\alpha) - C(\alpha_0))] - (\hat{K}(i\omega; \alpha) - \hat{K}(i\omega_0; \alpha_0))(\psi_1 + w, \psi^*)_{2\pi}

+ (G(\psi_1 + w, \omega, \alpha; \varepsilon), \psi^*)_{2\pi} = 0.\]

In what follows, we use a mapping \(H = (H_1, H_2)\) of \((w, \omega, \alpha, \varepsilon) \in X_1 \times \mathbb{R}^+ \times I \times \mathbb{R}\) to \(X_1 \times C\) with \(H_1\) defined by the left-hand side of (4.7) and with \(H_2\) by the left-hand side of (4.8). By virtue of (H.1)-(H.4), \(H\) is a \(C^*\)-mapping satisfying

\[H(0, \omega_0, \alpha_0; 0) = 0\]

(use (4.3)). To solve \(H(w, \omega, \alpha; \varepsilon) = 0\) in a neighborhood of \((0, \omega_0, \alpha_0, 0)\), the standard implicit function theorem will be useful (see, e.g., Crandall [3] or Sattinger [11, p. 60]).

We now take the Fréchet derivative of \(H = (H_1, H_2)\) with respect to \((w, \omega, \alpha)\) at \((w, \omega, \alpha, \varepsilon) = (0, \omega_0, \alpha_0, 0)\). By virtue of (4.3), we have

\[(4.9) \quad D_w H(0, \omega_0, \alpha_0; 0) = (1, 0).\]

Making use of (4.3), one obtains

\[(4.10) \quad D_w H_2(0, \omega_0, \alpha_0; 0) = i(\{[\mathrm{id} - D_2 \hat{K}(i\omega_0; \alpha_0)]\psi_1, \psi^*\}_{2\pi}

= 2i\langle[\mathrm{id} - D_2 \hat{K}(i\omega_0; \alpha_0)]a_\mu, a_\mu^*\rangle.\]
Similarly,
\begin{equation}
D_aH(0, \omega_0; \alpha_0; 0) \\
= (\mu D_\alpha D(\alpha_0) - D_\alpha C(\alpha_0) - D_\alpha \hat{K}(i\omega_0; \alpha_0)) \psi_1, \psi^* \eta \eta \eta \\
= 2(\mu D_\alpha D(\alpha_0) - D_\alpha C(\alpha_0) - D_\alpha \hat{K}(i\omega_0; \alpha_0)) \alpha_\delta, \alpha_\delta^*.
\end{equation}

In view of (4.9), (4.10) and (4.11), our hypotheses (H.6) and (H.7) assure that the Fréchet derivative of $H$ at $(0, \omega_0, \alpha_0, 0)$ is invertible. Hence, it follows from the implicit function theorem that there exists a unique family \((w(\epsilon), \omega(\epsilon), \alpha(\epsilon)) \in C'[(-\epsilon_0, \epsilon_0]; X_2 \times R^* \times I)\) with some $\epsilon_0 > 0$ such that
\begin{equation}
H(w(\epsilon), \omega(\epsilon), \alpha(\epsilon); \epsilon) = 0, \quad w(0) = 0, \quad \omega(0) = \omega_0, \quad \alpha(0) = \alpha_0.
\end{equation}

Hence the conclusion of Theorem I easily follows by noting (1.3), (1.4) and (4.1).

q.e.d.

4.2. Proof of Theorem II.

We first prepare the following lemma.

**Lemma 4.1.** Suppose that (H.1)–(H.3), (H.5) and (H.6) hold. Then there exist functions \((\lambda(\alpha), a(\alpha)) \in C'([\alpha_0 - \delta, \alpha_0 + \delta], C \times C^\times)\) with some $\delta > 0$ such that
\[
E_a(\lambda(\alpha), \alpha)a(\alpha) = 0, \quad \lambda(\alpha_0) = i\omega_0 \quad \text{and} \quad a(\alpha_0) = a_\delta.
\]

**Proof.** Set $E^0 = E_a(i\omega_0, \alpha_0)$. Since \(\text{dim Ker } E^0 = \text{codim Im } E^0 = 1\) (\(\text{Im } E^0 = \text{the image of } E^0\)), we can decompose
\[
C^\times = \text{Ker } E^0 \oplus Z_1 \quad \text{and} \quad C^\times = Z_0 \oplus \text{Im } E^0.
\]

Then the restriction $E^1$ of $E^0$ on $Z_1$ is invertible. We will solve the equation $E_a(\lambda, \alpha)a = 0$ in the form
\[
a = a_\delta + b \quad \text{with} \quad b \in Z_1.
\]

Then
\begin{equation}
E^1b + [E_a(\lambda, \alpha) - E^0](a_\delta + b) = 0.
\end{equation}

Since $\text{Im } E^0 = \{a \in C^\times; \langle a, a_\delta^* \rangle = 0 \text{ with } 0 \neq a_\delta^* \in \text{Ker } (E^0)^*\}$, one can get the equivalent form to (4.13): that is
\[
e_1(\lambda, b; \alpha) = \langle [E_a(\lambda, \alpha) - E^0](a_\delta + b), a_\delta^* \rangle = 0,
\]
\[
e_2(\lambda, b; \alpha) = b + (E^1)^{-1}Q_1[(E_a(\lambda, \alpha) - E^0)(a_\delta + b)] = 0,
\]
where $Q_1$ is the projection from $C^\times$ to $\text{Im } E^0$. Now the mapping $e = (e_1, e_2)$ is of class $C^1$ from $C \times Z_1 \times I$ to $C \times Z_1$ and satisfies $e(i\omega_0, 0; \alpha_0) = 0$. The Fréchet
derivative of \( e \) at \((\omega_0, 0, \alpha_0)\) is given by
\[
\begin{pmatrix} D_1e_1 & D_1e_1 \\ D_2e_2 & D_2e_2 \\ D_3e_3 & D_3e_3 \end{pmatrix}(\omega_0, 0; \alpha_0) = \begin{pmatrix} \langle D_2E_M(\omega_0, \alpha_0)\alpha_0, a_M^* \rangle \\ \langle D_0E_M(\omega_0, \alpha_0)\alpha_0, a_M^* \rangle \end{pmatrix} \neq 0.
\]
Since (H.6) implies
\[
\langle D_1E_M(\omega_0, \alpha_0)\alpha_0, a_M^* \rangle = \langle \text{Id} - D_1\tilde{K}(\omega_0; \alpha_0) \rangle a_M, a_M^* \neq 0,
\]
the assertion easily follows from the implicit function theorem. q.e.d.

We now proceed to the proof of Theorem II. It suffices to see that (H.7) is equivalent to (H.8). Let \( a(\alpha) \) and \( \lambda(\alpha) \) be the functions in Lemma 4.1. Differentiating
\[
\langle E_M(\lambda(\alpha), \alpha)\alpha(\alpha), a_M^* \rangle = 0, \quad a_M^* \in \text{Ker } E_M(\omega_0, \alpha_0)^*,
\]
with respect to \( \alpha \) and setting \( \alpha = \alpha_0 \), we obtain
\[
\langle D_1E_M(\omega_0, \alpha_0)\alpha_0, a_M^* \rangle D_\alpha \lambda(\alpha_0) + \langle D_0E_M(\omega_0, \alpha_0)\alpha_0, a_M^* \rangle \\
+ \langle E_M(\omega_0, \alpha_0)D_\alpha a(\alpha_0), a_M^* \rangle = 0.
\]
Since \( a_M^* \) is in \( \text{Ker } E_M(\omega_0, \alpha_0)^* \), it follows that
\[
D_\alpha \lambda(\alpha_0) = -\frac{\langle D_0E_M(\omega_0, \alpha_0)\alpha_0, a_M^* \rangle}{\langle D_1E_M(\omega_0, \alpha_0)\alpha_0, a_M^* \rangle}.
\]
The identity (4.14) gives the equivalence between (H.7) and (H.8). q.e.d.

**Remark 4.1.** When the convergence of \( \epsilon^{-1}D_\alpha F(\omega, \alpha) \) as \( \epsilon \to 0 \) in (H.4) (ii) is not assumed, the \( C^1 \)-dependence of \( H(\omega, \alpha; \epsilon) \) with respect to \( \epsilon \) is no longer satisfied. However, the implicit function theorem due to Crandall [3] assures the existence of a family \( (\omega(\epsilon), \alpha(\epsilon)) \in C([-\epsilon_0, \epsilon_0]; X \times \mathbb{R} \times \mathbb{R} \) satisfying (4.12). Therefore, the assertion of Remark 2.3 holds true.

§ 5. Applications

In this section we will apply the preceding results to semilinear Volterra diffusion equations arising in population dynamics. Let \( u \) denote the population density of a species which lives in \( \Omega \) undergoing diffusion and memory effects. We assume that the evolution of \( u \) is described by the following equations:
\[
\frac{\partial u}{\partial t} = du + u(a - bu - \int_{-\infty}^{\infty} k(t-s)u(s)ds) \quad \text{in } \Omega \times \mathbb{R},
\]
\[
\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times \mathbb{R},
\]
where $a > 0$, $b \geq 0$ and $d > 0$ are some ecological constants and $k$ is a $C^1$-function of $t$ such that $k, tk \in L^1(0, \infty)$. The Volterra integral in (5.1) means that the growth rate of $u$ depends on the past history of $u$. For the investigation of (5.1) in the case $d = 0$, we refer the reader to the monograph of Cushing [5], which contains various interesting results on the related problems.

In consideration of ecological meaning of our problem, we will consider only non-negative solutions of (5.1)–(5.2). If

$$\alpha \equiv \int_0^\infty k(t)dt > -b$$

is assumed, then it is well known that the stationary problem for (5.1)–(5.2) has a unique positive stationary solution

(5.3) \hspace{1cm} u^* \equiv a/(b + \alpha).

We will find out conditions under which bifurcation of periodic solutions occurs in a neighbourhood of $u = u^*$. By setting $v = u - u^*$, equations (5.1)–(5.2) are rewritten as follows:

(5.4) \hspace{1cm} \frac{\partial v}{\partial t} = dAt - bu^*v - u^* \int_{-\infty}^t k(t-s)\nu(s)ds + f(\nu) \quad \text{in } \Omega \times \mathbb{R},

(5.5) \hspace{1cm} \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times \mathbb{R},

where

(5.6) \hspace{1cm} f(\phi)(x) = -\phi(x, 0) \left( b\phi(x, 0) + \int_{-\infty}^0 k(-s)\phi(x, s)ds \right),

for $\phi : \Omega \times (-\infty, 0] \to \mathbb{R}$. Linearization of (5.4)–(5.5) leads us to the corresponding characteristic problem

(5.7) \hspace{1cm} (\lambda - dAt + u^*(b + k(\lambda)))w = 0 \quad \text{in } \Omega,

\hspace{1cm} \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial \Omega.

In terms of the eigenvalues $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots$ for (2.3) with $B = \partial/\partial \nu$, every characteristic value $\lambda$ (with $\text{Re } \lambda \geq 0$) of (5.7) is obtained as a complex number satisfying one of the following equations

(5.8) \hspace{1cm} E_m(\lambda) \equiv \lambda + d\mu_m + u^*(b + k(\lambda)) = 0, \quad \text{for } m = 0, 1, 2, \cdots.

In what follows, we will carry out the spectral analysis for (5.8) by taking some functions as $k$. 
Example 5.1. \( k(t) = \rho \alpha e^{-\rho t} \) with \( \alpha, \rho > 0 \). Since the Laplace transform of \( k \) is given by \( \hat{k}(\lambda) = \rho \alpha / (\lambda + \rho) \), it is easy to see that (5.7) has no characteristic values \( \lambda \) with \( \Re \lambda \geq 0 \). Therefore, no bifurcation of periodic solutions takes place. In fact, it is well known that every non-negative solution (not identically zero) for (5.1)–(5.2) converges to \( u^* \) uniformly on \( \Omega \) as \( \varepsilon \to 0 \). See Schiaffino [12] and Yamada [17].

Example 5.2. \( k(t) = \rho \alpha e^{-\rho t} \) with \( \alpha, \rho > 0 \). In this case,

\[
\hat{k}(\lambda) = \rho \alpha / (\lambda + \rho)^2;
\]

so that (5.8) is equivalent to

\[
\lambda^2 + (2\rho + bu^* + d\mu_m)\lambda^2 + \rho(\rho + 2(\rho + d\mu_m))\lambda + \rho^2((\alpha + b)u^* + d\mu_m) = 0
\]

for \( m = 0, 1, 2, \ldots \).

By virtue of Hurwitz criterion (see, e.g., [9, p. 141]), we see that, for each \( m \), (5.9) has no roots with \( \Re \lambda \geq 0 \) if and only if

\[
0 < \rho(\rho + 2(\rho + d\mu_m))(\rho + \rho^2((\alpha + b)u^* + d\mu_m)) - \rho^2(\rho^2((\alpha + b)u^* + d\mu_m) + 2\rho^3 + (4b - \alpha)u^* + 2b^2(u^*)^\gamma).
\]

For the sake of convenience, make only \( \alpha \) change as a parameter and other quantities \( b, u^* \) and \( \rho \) fixed. Set

\[
\alpha_0 = 2(\rho + bu^*)^\gamma / (u^* \rho).
\]

If \( \alpha < \alpha_0 \), then it follows from (5.10) that (5.7) has no characteristic values \( \lambda \) in \( \Re \lambda \geq 0 \). This fact implies the (local) asymptotic stability of \( u = 0 \) for (5.4)–(5.5), or equivalently, that of \( u = u^* \) for (5.1)–(5.2). For \( \alpha = \alpha_0 \), it is easy to verify (H.5) with \( E_m(\lambda; \alpha) = \lambda + d\mu_m + bu^* + \rho^2 \alpha u^*/(\lambda + \rho)^2 \), \( M = 0 \) and \( \omega_c = (\rho^2(\rho + 2bu^*))^{1/2} \). Furthermore, one can verify (H.6) and (H.7) (or (H.8)) after some calculations. Since the nonlinear term given by (5.6) satisfies (H.4), it follows from Theorem I (or Theorem II) that non-trivial periodic solutions of (5.1)–(5.2) bifurcate from \( u^* \). That is, as \( \alpha \) becomes greater than the critical value \( \alpha_0 \), the stationary solution \( u = u^* \) loses its stability and new periodic solutions appear near \( u^* \). (However, it should be observed that these bifurcating periodic solutions are spatially homogeneous because it is known that the same result holds for the case \( d = 0 \) in (5.1). See [5].)

We next set

\[
\alpha_m = 2(\rho + bu^* + d\mu_m)^\gamma / (u^* \rho),
\]

for every simple eigenvalue \( \mu_m \) (see (5.10)). Similarly as above, one can show that a pair of characteristic values of (5.7), which are pure imaginary for \( \alpha = \alpha_m \), satisfies the transversality condition (H.8). Hence, our bifurcation theory is applicable in this situation.
Remark 5.1. Tesei [16] has shown similar bifurcation results for (5.1)-(5.2) with the same kernel $k$ as ours. However, his method of proof is quite different from ours. Making use of the speciality of $k$, he has rewritten (5.1) in the form of a system of differential equations and applied the standard Hopf bifurcation theorem (see, e.g., Crandall and Rabinowitz [4]).

Example 5.3. $k(t) = (\alpha \rho^3 + 4\beta^2)/2\beta^2) e^{-\alpha t} \sin^2 \beta t$ with $\alpha, \beta, \rho > 0$. Since the Laplace transform of $k$ is
\[
\hat{k}(\lambda) = \frac{\alpha \rho^3 + 4\beta^2}{(\lambda + \rho)((\lambda + \rho)^3 + 4\beta^2)},
\]
(5.8) is written as follows:
\[
(5.11) \quad \lambda + bu^* + d\mu_m + \frac{\alpha \rho^3 + 4\beta^2}{(\lambda + \rho)((\lambda + \rho)^3 + 4\beta^2)} = 0, \quad m = 0, 1, 2, \ldots.
\]
If we set $\zeta = \lambda/\rho, \tau = 4\beta^2/\rho^3, B = bu^*/\rho, C = \alpha(1+\tau)u^*/\rho$ and $z_m = d\mu_m/\rho$, then (5.11) can be rewritten as
\[
(5.12) \quad \zeta^4 + (B + z_m + 3)\zeta^4 + (\tau + 3 + 3(B + z_m))\zeta^2 + [(\tau + 1) + (B + z_m)(\tau + 3)]\zeta
\]
\[
+ (B + z_m)(\tau + 1) + C = 0,
\]
for $m = 0, 1, 2, \ldots$. As in Example 5.2, Hurwitz criterion assures that, for every $m$, (5.12) has no roots $\zeta$ in the plane $\text{Re } \zeta \geq 0$ if and only if
\[
(5.13) \quad C < 2(\tau + 4)\Phi(B + z_m),
\]
where
\[
\Phi(B) = (B + 1)(B^2 + 2B + \tau + 1)/(B + 3)^4.
\]
In the case when (5.13) is satisfied for every $m$, the stationary solution $u^*$ is asymptotically stable for (5.1)-(5.2). However, we should observe that, even if $u^*$ is asymptotically stable as the solution of the ordinary functional differential equation ($d=0$), $u^*$ is not necessarily asymptotically stable as the solution of the partial functional differential equation ($d>0$). That is, in some cases, (5.13) holds for $m=0$, but it does not hold for some $m \neq 0$. For example, suppose
\[
B > 1 \quad \text{and} \quad \tau > (B+1)^2(B+7)/(B-1);
\]
in other words,
\[
bu^* > \rho \quad \text{and} \quad 4\beta^2 > \frac{(bu^* + \rho)^2}{(bu^* - \rho)}(bu^* + 7\rho).
\]
After some calculations, one has
\[
\Phi'(B) = (B + 3)^{-3}[(B + 1)^3(B + 7) - \tau(B - 1)] < 0,
\]
so that
\[
\min_{m} \Phi(B + z_m) = \Phi(B + z_N) \quad \text{for some } z_N = d\mu/\rho \neq 0.
\]
(Make \(d\mu/\rho\) enough small if necessary.) Therefore, provided that \(C(=\alpha(1 + \tau)u^*/\rho)\) is smaller than \(\Phi_0 \equiv 2(\tau + 4)\Phi(B + z_N)\), then \(u^*\) is asymptotically stable. However, as \(C\) becomes greater than \(\Phi_0\), \(u^*\) loses its stability. This is the case to which our bifurcation theory is applicable and periodic solutions, which are spatially inhomogeneous, appear as the primary bifurcation.

**Remark 5.2.** Yoshida and Kishimoto [20] have studied semilinear diffusion equations with two time-delays and shown that, under some conditions, a bifurcation of spatially inhomogeneous periodic solutions occurs as a primary one. Similar phenomenon is shown in Example 5.3, where we may consider that the kernel function \(k\) has many number of time delays in a sense.

In contrast to the above situation, it is shown by Yoshida [19] that semilinear diffusion equations with only one time-delay always exhibit bifurcations of spatially homogeneous periodic solutions as primary ones (cf. Example 5.2).

**Remark 5.3.** In a similar manner to this section, we can apply our bifurcation theory to a system of semilinear parabolic equations with infinite delays such as prey-predator systems with memory and diffusion effects.

**Appendix.**

For the sake of completeness, we will show the following result.

**Proposition A.** The operator \(\bar{L}_0\) defined by (3.13) is closed in \(H_{2\alpha}\). The adjoint operator \(\bar{L}_0^*\) of \(\bar{L}_0\) is given by

\[
(\bar{L}_0^*v)(s) = -\omega_v \frac{\partial}{\partial s}v(s) - D(\alpha_0)\Delta v(s) - C(\alpha_0)^*v(s)
\]

(A.1)

\[-\int_s^\infty K(\sigma - s; \omega_0, \omega_0)v(\sigma)d\sigma,
\]

with domain \(D(\bar{L}_0^*) = D(\bar{L}_0)\), where \(C^*\) and \(K^*\) denote the adjoint matrices of \(C\) and \(K\).

Proposition A will be proved with the aid of a series of Lemmas. First we define \(\bar{J}_0\) in \(H_{2\alpha}\) by

\[
\bar{J}_0u = \omega_0 \frac{\partial}{\partial s}u - D(\alpha_0)\Delta u - C(\alpha_0)u
\]
with domain \( D(\tilde{J}_0) = D(\tilde{L}_0) \). Then we have

**Lemma A.1.** Let \( g \in H_{2\pi} \) be given. If \( \zeta \) is sufficiently large, then there exists a unique \( u \in D(\tilde{J}_0) \) satisfying \( (\tilde{J}_0 + \zeta)u = g \). Moreover,

\[
\sum_{i=1}^{\infty} |D^i u|_{L^2}^2 + \left| \frac{\partial u}{\partial s} \right|_{L^2}^2 \leq C(\zeta) |g|_{L^2}^2,
\]

with some \( C(\zeta) > 0 \).

**Proof.** It suffices to repeat the similar arguments to those in the proof of Lemma 3.1 by making use of some appropriate Sobolev spaces in place of function spaces of Schauder type. q.e.d.

By Lemma A.1, it is easy to see that \( \tilde{J}_0 \) is a closed linear operator with dense domain in \( H_{2\pi} \). As to the adjoint operator \( \tilde{J}_0^* \) of \( \tilde{J}_0 \), we have

**Lemma A.2.** \( \tilde{J}_0^* \) is given by

\[
(\tilde{J}_0^* v) = -\omega_0 \frac{\partial v}{\partial s} - D(\alpha_0) D v - C(\alpha_0)^* v
\]

with domain \( D(\tilde{J}_0^*) = D(\tilde{J}_0) \).

**Proof.** We denote by \( \tilde{J}_1 \) the operator defined by the right-hand side of (A.2) with \( D(\tilde{J}_1) = D(\tilde{J}_0) \). For \( u, v \in D(\tilde{J}_0) \),

\[
(\tilde{J}_0 v, u)_{L^2} = (u, \tilde{J}_1 v)_{L^2},
\]

which implies \( \tilde{J}_1 \subset \tilde{J}_0^* \). Moreover, one can show in the same way as Lemma A.1 that, if \( \zeta \) is sufficiently large, then \( \zeta \) belongs to the resolvent set of \( -\tilde{J}_1 \) (which is denoted by \( \rho(-\tilde{J}_1) \)).

Let \( v \) be any element in \( D(\tilde{J}_0^*) \). Then there exists a unique element \( v_1 \in D(\tilde{J}_1) \) such that

\[
(\tilde{J}_0^* + \zeta)v = (\tilde{J}_1 + \zeta)v_1 \quad \text{if} \quad \zeta \in \rho(-\tilde{J}_1).
\]

Hence

\[
(\tilde{J}_0^* + \zeta)(v - v_1) = 0.
\]

It is well-known that \( \zeta \in \rho(-\tilde{J}_1) \) if and only if \( \zeta \in \rho(-\tilde{J}_0^*) \) (see Yosida [21; p. 225]); so that (A.3) implies \( v = v_1 \in D(\tilde{J}_1) \). Thus we have shown \( \tilde{J}_1 = \tilde{J}_0^* \). q.e.d.

Next we define

\[
(\bar{K}_0 u)(s) = \int_{-\infty}^{\infty} K(s - \sigma; \omega_0, \alpha_0) u(\sigma) d\sigma \quad \text{for} \quad u \in H_{2\pi}.
\]
Lemma A.3. \( \tilde{K}_0 \) is a bounded linear operator in \( H_{x^*} \) such that

\[
|\tilde{K}_0 u|_{L^2} \leq \|\kappa\|_{L^1(0,\infty;R)} |u|_{L^2} \quad \text{for } u \in H_{x^*},
\]

where \( \kappa \) is the integrable function in (H.3). The adjoint operator \( \tilde{K}_0^* \) of \( \tilde{K}_0 \) is given by

\[
(\tilde{K}_0^* v)(s) = \int_s^{\infty} K(s-\tau; \omega_0, \alpha_0) v(\tau) d\tau \quad \text{for } v \in H_{x^*}.
\]

Proof. For \( u \in H_{x^*} \), we have

\[
2\pi |\tilde{K}_0 u|_{L^2}^2 = \int_0^{2\pi} \left( \int_{-\infty}^{\infty} K(s-\sigma; \omega_0, \alpha_0) v(\sigma) d\sigma \right)^2 ds
\]

\[
\leq \int_0^{2\pi} \left( \int_{-\infty}^{\infty} |K(s-\sigma; \omega_0, \alpha_0)| \cdot \|v(\sigma)\|_{L^2(\mathbb{R}; C^\infty)} d\sigma \right)^2 ds
\]

\[
\leq \int_0^{2\pi} \left( \int_{-\infty}^{\infty} |K(s-\sigma; \omega_0, \alpha_0)| d\sigma \int_{-\infty}^{\infty} |K(s-\sigma; \omega_0, \alpha_0)| \cdot \|v(\sigma)\|_{L^2(\mathbb{R}; C^\infty)} d\sigma \right) ds
\]

\[
= \|\kappa\|_{L^1(0,\infty;R)} \int_0^{2\pi} \int_{-\infty}^{\infty} |K(s-\sigma; \omega_0, \alpha_0)| \cdot \|v(\sigma)\|_{L^2(\mathbb{R}; C^\infty)} d\sigma ds.
\]

Using the 2\( \pi \)-periodicity of \( u \), one can also get

\[
\int_{-\infty}^{\infty} |K(s-\sigma; \omega_0, \alpha_0)| \cdot \|v(\sigma)\|_{L^2(\mathbb{R}; C^\infty)} d\sigma ds
\]

\[
= \left( \int_0^{2\pi} d\sigma + \sum_{j=0}^{\infty} \left( \int_0^{2\pi} d\sigma + \int_{-2\pi j+\pi}^{-2\pi j+2\pi} d\sigma \right) \right) |K(s-\sigma; \omega_0, \alpha_0)| \cdot \|v(\sigma)\|_{L^2(\mathbb{R}; C^\infty)} ds
\]

\[
= \int_0^{2\pi} |K(s-\sigma; \omega_0, \alpha_0)| \cdot \|v(\sigma)\|_{L^2(\mathbb{R}; C^\infty)} ds
\]

\[
+ \sum_{j=0}^{\infty} \left( \int_0^{2\pi} d\sigma + \int_{-2\pi j+\pi}^{-2\pi j+2\pi} d\sigma \right) |K(s+2(j+1)\pi-\sigma; \omega_0, \alpha_0)| \cdot \|v(\sigma)\|_{L^2(\mathbb{R}; C^\infty)} d\sigma
\]

\[
= \int_0^{2\pi} \|v(\sigma)\|_{L^2(\mathbb{R}; C^\infty)} ds \int_0^{2\pi} |K(s-\sigma; \omega_0, \alpha_0)| ds
\]

\[
\leq \|\kappa\|_{L^1(0,\infty;R)} \int_0^{2\pi} \|v(\sigma)\|_{L^2(\mathbb{R}; C^\infty)} d\sigma.
\]

Hence (A.6) and (A.7) yield (A.4).

For \( u, v \in H_{x^*} \), one has

\[
2\pi (\tilde{K}_0 u, v)_{x^*} = \int_0^{2\pi} d\sigma \left( \int_{-\infty}^{\infty} K(s-\sigma; \omega_0, \alpha_0) u(\sigma) d\sigma, v(s) \right)_{L^2(\mathbb{R}; C^\infty)}.
\]

As in the calculations of (A.7), we use the 2\( \pi \)-periodicity of \( u \) and change the order of integration; then

\[
2\pi (\tilde{K}_0 u, v)_{x^*} = \int_0^{2\pi} d\sigma \int_{-\infty}^{\infty} (K(s-\sigma; \omega_0, \alpha_0) u(\sigma), v(s))_{L^2(\mathbb{R}; C^\infty)} ds
\]
Thus (A.5) is derived.

We are now in a position to accomplish the proof of Proposition A. Note that \( \overline{L}_0 \) is expressed as follows:

\[
\overline{L}_0 = \overline{J}_0 - \overline{K}_0.
\]

Since \( \overline{J}_0 \) is closed in \( H_{2e} \) and \( \overline{K}_0 \) is bounded in \( H_{2e} \) (by (A.4)), it is easy to see that \( \overline{L}_0 \) is closed in \( H_{2e} \). To get the expression (A.1) for \( \overline{L}_{2e} \), it is sufficient to note the boundedness of \( \overline{K}_0 \) and use (A.2) and (A.5).

References


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