
MULTIPLE EXISTENCE OF POSITIVE SOLUTIONS OF COMPETING SPECIES EQUATIONS WITH DIFFUSION AND LARGE INTERACTIONS

Dedicated to Professors Masayasu Mimura and Takaaki Nishida on the occasion of their sixtieth birthday

Takefumi Hirose and Yoshio Yamada

Abstract. This paper is concerned with positive solutions of Lotka-Volterra competition system with diffusion

\[
\begin{align*}
-u'' &= u(a_1 - u - b_1v), \quad 0 < x < 1, \\
-v'' &= v(a_2 - v - b_2u), \quad 0 < x < 1, \\
\end{align*}
\]

\( (P) \)

where \(a_1, a_2, b_1, b_2\) are positive constants. When \(b_1\) and \(b_2\) are sufficiently large, there is a close relationship between \((P)\) and the corresponding limit problem. Complete information on the set of solutions for the limit problem yields multiple existence of positive solutions of \((P)\) and their instability properties for sufficiently large \(b_1, b_2\).

1 Introduction

This paper is concerned with the following system of differential equations:

\[
\begin{align*}
-u'' &= u(a_1 - u - b_1v) \quad \text{for } 0 < x < 1, \\
-v'' &= v(a_2 - v - b_2u) \\
\end{align*}
\]

\( (1.1) \)

\( \text{for } 0 < x < 1, \\
\end{align*}
\]

\( u(0) = u(1) = v(0) = v(1) = 0, \)

\( \)

\( ^1\)Partially supported by Waseda University Grant for Special Projects 2000A-144
where $a_i, b_i$ ($i = 1, 2$) are strictly positive constants. We study the multiple existence of positive solutions of (1.1) and their profiles when $b_1$ and $b_2$ are sufficiently large. This system is a one-dimensional version of the following Lotka-Volterra competition system with diffusion:

$$\begin{align*}
-\Delta u &= u(a_1 - u - b_1 v) \quad \text{in } \Omega, \\
-\Delta v &= v(a_2 - v - b_2 u) \quad \text{in } \Omega, \\
u = v &= 0 \quad \text{on } \partial \Omega,
\end{align*}$$

(1.2)

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with smooth boundary $\partial \Omega$. In (1.2), $u$ and $v$ represent population densities of two competing species, $a_1, a_2$ denote the intrinsic growth rates and $b_1, b_2$ denote the interspecific competition rates between two species.

Positive solutions for (1.2) have been studied by many authors and various methods have been developed to construct positive solutions (see, e.g., Blat-Brown [1], Dancer [2], [3], [4], López-Gómez-Pardo [12] and the references therein). However, the information on the set of positive solutions is still far from complete and it is not easy to get satisfactory results on the uniqueness or non-uniqueness of positive solutions. For the multiple existence of positive solutions we refer to [4], [10] and [11].

When $b_1$ and $b_2$ in (1.2) go to infinity with $b_2/b_1 \to \alpha$, Dancer and Du [5] have established a very interesting relationship between (1.2) and the following limit problem:

$$\begin{align*}
-\Delta w &= w^+ (a_1 - w^+ / \alpha) + w^- (a_2 + w^-) \quad \text{in } \Omega, \\
w &= 0 \quad \text{on } \partial \Omega.
\end{align*}$$

(1.3)

In fact, if $w_0$ is an isolated solution of (1.3) such that $w_0$ changes sign in $\Omega$ and has non-zero index, then there exist positive solutions $(u, v)$ of (1.2) such that $(u, v) \to (w_0^+ / \alpha, -w_0^-)$ in $L^2(\Omega) \times L^2(\Omega)$ as $b_1, b_2 \to \infty$, where $w^+ = \max\{w, 0\}$ and $w^- = \min\{w, 0\}$. It also follows from Dancer and Guo [7] that, for sufficiently large $b_1, b_2$ with $b_2/b_1$ close to $\alpha$, a positive solution of (1.2) is unique in a neighborhood of $(w_0^+ / \alpha, -w_0^-)$ and that the positive solution is stable (resp. unstable) if $w_0$ is a stable (resp. unstable) solution of (1.3). By the stability we mean the stability as solutions of the corresponding parabolic equations. Thus the analysis of changing sign solutions of (1.3) gives us very useful information on positive solutions of (1.2) for sufficiently large $b_1$ and $b_2$.

Similar result also holds true when $b_2/b_1 \to \infty$ as $b_1, b_2 \to \infty$. The corresponding limit problem is given by

$$\begin{align*}
-\Delta w &= a_1 w^+ + w^- (a_2 + w^-) \quad \text{in } \Omega, \\
w &= 0 \quad \text{on } \partial \Omega.
\end{align*}$$

(1.4)

The purpose of this paper is to give complete results on the structure of solutions of (1.3) and (1.4) in the one-dimensional case, and to give much better understanding on positive solutions of (1.1) when $b_1, b_2$ are sufficiently large. We study (1.3) and (1.4) by the usual phase-plane method and calculate the indices of their solutions. When $b_1$ and $b_2$ are sufficiently large, we show that (1.1) has finitely many positive solutions and that the number of solutions depends on the location of $a_1, a_2$ in $a_1-a_2$ plane. Moreover, it is also shown that, if $b_1$ and $b_2$ are sufficiently large, the profiles of positive solutions of (1.1) are quite similar to those of the corresponding changing sign solutions of (1.3) (or (1.4)).

In Section 2 we state some important preliminary results established in [5] and [7]. In Section 3 we discuss limit equations (1.3) and (1.4). Section 4 is devoted to the study
of profiles of positive solutions of (1.1). In Section 5 we briefly discuss the corresponding non-stationary problem and related problems.

2 Preliminary Theorems

It is well known that (1.1) (or (1.2)) has a positive solution only if \(a_1 > \lambda_1\) and \(a_2 > \lambda_1\) (see [1] or [2]). Here we denote by \(\{\lambda_n\}\) the eigenvalues of \(-\Delta\) with homogeneous Dirichlet boundary condition and they satisfy \(0 < \lambda_1 < \lambda_2 \leq \cdots\). 

First of all, using the maximum principle we can show the following lemma.

**Lemma 2.1.** Any positive solution \((u, v)\) of (1.2) satisfies
\[
0 < u \leq a_1, \quad 0 < v \leq a_2 \quad \text{in } \Omega. \tag{2.1}
\]

We know that positive solutions of (1.2) are generated by limiting problems (1.3). The following result due to Dancer-Du [5, Theorem 3.3] and Dancer-Guo [7, Theorem 1.2] plays a fundamental role in the present paper.

**Theorem 2.2.** Suppose that \(w_0\) is an isolated solution of (1.3) which changes sign in \(\Omega\) and has non-zero index. Then for any \(\varepsilon > 0\), there exist positive constant \(\overline{b}\) and \(\delta\) such that for any \(b_1, b_2\) with
\[
b_1 > \overline{b}, \quad |b_2/b_1 - \alpha| < \delta,
\]
(1.2) has a unique positive solution \((u, v)\) such that
\[
\|u - w_0^+ / \alpha\|_p < \varepsilon \quad \text{and} \quad \|v + w_0^-\|_p < \varepsilon.
\]

Here \(\| \cdot \|_p\) denotes the \(L^p(\Omega)\)-norm with \(p > \max\{2, N/2\}\) and the index of \(w_0\) means the fixed point index:
\[
\text{index}_{C^0(\overline{\Omega})}(A_{\alpha}, w_0),
\]
where
\[
A_{\alpha}(w) = (-\Delta)^{-1} \left( w^+ \left( a_1 - w^+ / \alpha \right) + w^- (a_2 + w^-) \right)
\]
and \(C^0(\overline{\Omega})\) denotes the space of \(C^1(\overline{\Omega})\)-functions vanishing on \(\partial\Omega\).

**Remark 2.1.** In [6], it is proved that (1.3) admits at least one changing sign solution for \(a_1 > \lambda_2\) and \(a_2 > \lambda_2\) and that, if each non-trivial solution is isolated, then (1.3) admits a changing sign solution whose index is \(-1\). Moreover, they also have obtained necessary and sufficient conditions for the existence of changing sign solutions. Actually, in \(a_1-a_2\) plane with \(a_1, a_2 > \lambda_1\), there exists a continuous and strictly decreasing curve \(\Gamma\) passing through \((\lambda_2, \lambda_2)\) such that (1.3) (resp. (1.4)) has a changing sign solution if and only if \((a_1, a_2)\) is located above \(\Gamma\) in \(a_1-a_2\) plane.

The following theorem is concerned with the case \(\alpha = +\infty\) in (1.3). See [5, Theorem 3.4] and [7, Theorem 1.3].
**Theorem 2.3.** Suppose that \( u_0 \) is an isolated solution of (1.4) which changes sign in \( \Omega \) and has non-zero index. Then for any \( \varepsilon > 0 \) there exist positive constants \( \bar{b} \) and \( M \) such that for any \( b_1, b_2 \) satisfying \( b_1 > \bar{b} \) and \( b_2/b_1 > M \), (1.2) admits a positive solution \( (u, v) \) such that
\[
\|b_2u/b_1 - w_0^+\|_p < \varepsilon \quad \text{and} \quad \|v + w_0^-\|_p < \varepsilon.
\]
Here the index of \( w_0 \) means \( \text{index}_{C_0(\overline{\Omega})} (A_{\infty}, w_0) \) with
\[
A_{\infty}(w) = (-\Delta)^{-1}(a_1 w^+ + w^-(a_2 + w^-)).
\]

**Remark 2.2.** Dancer and Guo [7] also have discussed the stability of positive solutions of (1.2) constructed in Theorems 2.2 and 2.3. Let \( u_0 \) be a non-degenerate solution of (1.3) or (1.4) which changes sign in \( \Omega \). For sufficiently large \( b_1, b_2 \) with \( b_2/b_1 \) close to \( \alpha \), let \( (u, v) \) be a unique positive solution of (1.2) near to \( (w_0^+ /\alpha, -w_0^-) \) in \( L^p(\Omega) \times L^p(\Omega) \) with \( p > \max\{2, N/2\} \). It is proved that \( (u, v) \) is stable (resp. unstable) if \( u_0 \) is a stable (resp. unstable) solution of (1.3). In Section 5 we will show the instability of every sign changing solution of (1.3) for the case \( N = 1 \); so that positive solutions of (1.2) become unstable for sufficiently large \( b_1, b_2 \) (see also [8]).

The similar results are also valid for (1.4).

3 Analysis of Limit Equations

In this section, we will give complete information on changing sign solutions of (1.3) and (1.4) in the one-dimensional case.

We begin with the analysis of (1.3) with \( N = 1 \) and \( \Omega = (0, 1) \):
\[
\begin{aligned}
- u'' &= g(u) & \text{for } 0 < x < 1, \\
w(0) &= w(1) = 0,
\end{aligned}
\]

where
\[
g(u) = w^+ (a_1 - w^+/\alpha) + w^- (a_2 + w^-)
\]
with \( (a_1, a_2) \in (0, \infty) \times (0, \infty) \). All nontrivial solutions of (3.1) are given by the following theorem.

**Theorem 3.1.** (i) If \( a_1 \leq \pi^2 \) and \( a_2 \leq \pi^2 \), then (3.1) admits no nontrivial solutions.
(ii) If \( a_1 \) and \( a_2 \) satisfy
\[
\frac{k}{\sqrt{a_1}} + \frac{k}{\sqrt{a_2}} < \frac{1}{\pi}
\]
for some positive integer \( k \), then (3.1) has exactly two nontrivial solutions which have \( 2k - 1 \) zero points in \( (0, 1) \); one satisfies \( w'(0) > 0 \) and the other \( w'(0) < 0 \).
(iii) If \( a_1 \) and \( a_2 \) satisfy
\[
\frac{k + 1}{\sqrt{a_1}} + \frac{k}{\sqrt{a_2}} < \frac{1}{\pi} \quad \left( \text{resp.} \quad \frac{k}{\sqrt{a_1}} + \frac{k + 1}{\sqrt{a_2}} < \frac{1}{\pi} \right)
\]
for some nonnegative integer \( k \), then (3.1) has a unique nontrivial solution which has \( 2k \) zero points in \( (0, 1) \) and satisfies \( w'(0) > 0 \) (resp. \( w'(0) < 0 \)).
Proof. The proof is based on the standard phase plane analysis. See, e.g., the monograph of Smoller [15] (see also [9]).

We will give the outline of the proof. Consider the following initial value problem:

\[
\begin{cases}
-w'' = g(w) & \text{for } x > 0, \\
w(0) = 0, & w'(0) = \gamma \in \mathbb{R},
\end{cases}
\]  

(3.2)

which has a unique solution \( w(x, \gamma) \) for each \( \gamma \). Our task is to find an appropriate \( \gamma \in \mathbb{R} \) satisfying \( w(1, \gamma) = 0 \); then \( w(x, \gamma) \) becomes a solution of (3.1).

Multiplying the both sides of (3.2) by \( w' \) and integrating the resulting expression from 0 to \( x \) we get

\[
\frac{1}{2} w'(x, \gamma)^2 + G(w(x, \gamma)) = \frac{1}{2} \gamma^2,
\]

(3.3)

where

\[
G(w) = \int_0^w g(v)dv = \begin{cases}
\frac{a_1 w^2}{2} - \frac{w^3}{3}\alpha & \text{if } w \geq 0, \\
\frac{a_2 w^2}{2} + \frac{w^3}{3} & \text{if } w \leq 0.
\end{cases}
\]

For each \( \gamma \in (0, \gamma_0) \) with \( \gamma_0 = \alpha a_1^{3/2}/\sqrt{3} \), define

\[
I_1(\gamma) = \sup \{ \ell > 0; w(x, \gamma) > 0 \text{ for } 0 < x < \ell \}.
\]

Usually, \( I_1(\gamma) \) is called a “time-map” for positive solutions of (3.2). Making use of (3.3) one can prove that \( I_1(\gamma) \) is a strictly increasing function of class \( C^1(0, \gamma_0) \) such that

\[
I_1(\gamma) \rightarrow \frac{\pi}{\sqrt{a_1}} \quad \text{as } \gamma \downarrow 0 \quad \text{and} \quad I_1(\gamma) \rightarrow \infty \quad \text{as } \gamma \uparrow \gamma_0.
\]

Similarly, we define a time-map \( I_2(\gamma) \) for negative solutions of (3.2) by

\[
I_2(\gamma) = \sup \{ \ell > 0; w(x, -\gamma) < 0 \text{ for } 0 < x < \ell \}.
\]

Set \( \overline{\gamma}_0 = a_2^{3/2}/\sqrt{3} \). We can also show that \( I_2(\gamma) \) is a strictly increasing function of class \( C^1(0, \overline{\gamma}_0) \) such that

\[
I_2(\gamma) \rightarrow \frac{\pi}{\sqrt{a_2}} \quad \text{as } \gamma \downarrow 0 \quad \text{and} \quad I_2(\gamma) \rightarrow \infty \quad \text{as } \gamma \uparrow \overline{\gamma}_0.
\]

If \( a_1 \leq \pi^2 \) and \( a_2 \leq \pi^2 \), then \( I_1(\gamma) > 1 \) and \( I_2(\gamma) > 1 \) for all \( \gamma \in (0, \min\{\gamma_0, \overline{\gamma}_0\}) \); so that there exists no real number \( \gamma \neq 0 \) such that \( w(1, \gamma) = 0 \). This fact implies that (3.2) has no nontrivial solutions.

In order to prove (ii), assume \( k(1/\sqrt{a_1} + 1/\sqrt{a_2}) < 1/\pi \). In view of the above properties of \( I_1 \) and \( I_2 \), it is seen that there exists a unique \( \gamma_k \in (0, \min\{\gamma_0, \overline{\gamma}_0\}) \) such that

\[
k I_1(\gamma_k) + k I_2(\gamma_k) = 1.
\]

Clearly, \( w(x, \gamma_k) \) is a unique solution of (3.1) with precisely \( 2k - 1 \) zero points in \((0,1)\) and \( w'(0, \gamma_k) > 0 \). Note that \( w(x, -\gamma_k) \) is also another solution of (3.1).

The proof of (iii) is almost the same; so we omit it.\]
Remark 3.1. Let
\[
C_0 = \left\{ (a_1, a_2) \in \mathbb{R}^2; \quad \frac{k_1}{\sqrt{a_1}} + \frac{k_2}{\sqrt{a_2}} = \frac{1}{\pi} \text{ for some nonnegative integers } (k_1, k_2) \neq (0, 0) \text{ satisfying } |k_1 - k_2| \leq 1 \right\}.
\]

One can show that all nontrivial solutions of (3.1) bifurcate from the trivial solution \( w = 0 \) at \((a_1, a_2) \in C_0\).

For \((a_1, a_2) \in (0, \infty) \times (0, \infty)\) we define
\[
D_k^1 = \left\{ (a_1, a_2); \quad \frac{k}{\sqrt{a_1}} + \frac{k}{\sqrt{a_2}} < \frac{1}{\pi}, \quad \frac{k+1}{\sqrt{a_1}} + \frac{k}{\sqrt{a_2}} \geq \frac{1}{\pi}, \quad \frac{k}{\sqrt{a_1}} + \frac{k+1}{\sqrt{a_2}} \geq \frac{1}{\pi} \right\},
\]
\[
D_k^2 = \left\{ (a_1, a_2); \quad \frac{k+1}{\sqrt{a_1}} + \frac{k}{\sqrt{a_2}} \geq \frac{1}{\pi}, \quad \frac{k+1}{\sqrt{a_1}} + \frac{k}{\sqrt{a_2}} < \frac{1}{\pi}, \quad \frac{k}{\sqrt{a_1}} + \frac{k+1}{\sqrt{a_2}} < \frac{1}{\pi} \right\},
\]
\[
D_k^3 = \left\{ (a_1, a_2); \quad \frac{k+1}{\sqrt{a_1}} + \frac{k}{\sqrt{a_2}} \geq \frac{1}{\pi}, \quad \frac{k+1}{\sqrt{a_1}} + \frac{k}{\sqrt{a_2}} \leq \frac{1}{\pi} \right\},
\]
\[
D_k^4 = \left\{ (a_1, a_2); \quad \frac{k+1}{\sqrt{a_1}} + \frac{k}{\sqrt{a_2}} \leq \frac{1}{\pi}, \quad \frac{k+1}{\sqrt{a_1}} + \frac{k}{\sqrt{a_2}} \geq \frac{1}{\pi} \right\},
\]
where \(k\) is nonnegative integer. See Figure 1. Obviously,
\[
(0, \infty) \times (0, \infty) = \bigcup_{k=0}^{\infty} (D_k^1 \cup D_k^2 \cup D_k^3 \cup D_k^4).\]

Furthermore, let \(w_{k,+}\) (resp. \(w_{k,-}\)) denote the solution of (3.1) which has precisely \(k\)-zero points in \((0, 1)\) and satisfies \(w'(0) > 0\) (resp. \(w'(0) < 0\)). Recall from the proof of Theorem 3.1 that these solutions are uniquely determined whenever they exist. Therefore, we have

Corollary 3.2. Define \(W = \{ w \in C^2[0, 1]; w \text{ is a solution of (3.1)} \}\). Then it holds that
\[
W = \begin{cases} 
\{0, w_{0,\pm}, w_{1,\pm}, \cdots, w_{2k-1,\pm}\} & \text{if } (a_1, a_2) \in D_k^1, \\
\{0, w_{0,\pm}, w_{1,\pm}, \cdots, w_{2k,\pm}\} & \text{if } (a_1, a_2) \in D_k^2, \\
\{0, w_{0,\pm}, w_{1,\pm}, \cdots, w_{2k-1,\pm}, w_{2k,\pm}\} & \text{if } (a_1, a_2) \in D_k^3, \\
\{0, w_{0,\pm}, w_{1,\pm}, \cdots, w_{2k-1,\pm}, w_{2k,-}\} & \text{if } (a_1, a_2) \in D_k^4.
\end{cases}
\]

In particular, every element of \(W\) is isolated.

We will calculate \(\text{index}_{C^0[0,1]}(A_\alpha, w)\) of \(w \in W\), where \(A_\alpha(w) = (-\Delta)^{-1}g(w)\) as a next step.

Theorem 3.3. (i) Each nontrivial solution \(w_{k,\pm}\), \(k = 0, 1, 2, \cdots\), of (3.1) satisfies
\[
\text{index}_{C^0[0,1]}(A_\alpha, w_{k,\pm}) = (-1)^k.
\]
(ii) The trivial solution of (3.1) satisfies
\[
\text{index}_{C^1_0[0,1]}(A_\alpha, 0) = \begin{cases} 
1 & \text{if } (a_1, a_2) \in D_k^1, \\
-1 & \text{if } (a_1, a_2) \in D_k^2, \\
0 & \text{if } (a_1, a_2) \in D_k^0 \cup D_k^1.
\end{cases}
\]

Before proving Theorem 3.3, we will state the following result due to Dancer and Du, whose proof can be found in [5, pp. 463-465].

**Lemma 3.4.** (i) For sufficiently large \( R \) it holds that
\[
\text{deg}_{C^1_0(\overline{\Omega})}(I - A_\alpha, B_R, 0) = 1,
\]
where \( B_R \) is a ball in \( C^1_0(\overline{\Omega}) \) with center 0 and radius \( R \).

(ii) For every \( a_1 > \pi^2 \)
\[
\text{index}_{C^1_0(\overline{\Omega})}(A_\alpha, u_{0,+}) = 1
\]
and for every \( a_2 > \pi^2 \)
\[
\text{index}_{C^1_0(\overline{\Omega})}(A_\alpha, u_{0,-}) = 1.
\]

**Remark 3.2.** By the maximum principle for elliptic equations, every solution \( w \) of (3.1) satisfies
\[-a_2 \leq w \leq a_1 \alpha \quad \text{in} \quad (0, 1),\]
which implies
\[-\frac{a_2^2}{4} \leq -w'' = g(w) \leq \frac{a_1^2 \alpha}{4}.\]

Since
\[
\| w_x \|_{\infty} \leq \| w_{xx} \|_{\infty},
\]
where \( \| \cdot \|_{\infty} = \sup \{ |v(x)| : 0 < x < 1 \} \). Every \( w \) in \( W \) satisfies
\[
\| w \|_{C^1_0[0,1]} = \| w \|_{\infty} + \| w_x \|_{\infty} \leq \max\{a_1 \alpha, a_2\} + \max\{\frac{a_1^2 \alpha}{4}, \frac{a_2^2}{4}\}. \quad (3.4)
\]

In Lemma 3.4 it is sufficient to choose sufficiently large \( R \) which is larger than the right-hand side of (3.4).

**Proof of Theorem 3.3.** We first consider the case \( a_1 = a_2 = a \). Then \( A_\alpha(w) \) is continuously differentiable with respect to \( w \); so that one can apply the Leray-Schauder index formula to get
\[
\text{index}_{C^1_0[0,1]}(A_\alpha, 0).
\]
Observe that the Fréchet derivative of \( A_\alpha \) at \( w = 0 \) is given by \( A'_\alpha(0)\tilde{w} = a (-d^2/dx^2)^{-1} \tilde{w} \). The corresponding eigenvalue problem
\[
\begin{cases}
-\lambda \tilde{w}'' = a \tilde{w} \\
\tilde{w}(0) = \tilde{w}(1) = 0
\end{cases} \quad \text{for } 0 < x < 1 \quad (3.5)
\]
has simple eigenvalues $a/(n^2\pi^2)$ $n = 1, 2, 3, \cdots$. Hence the Leray-Schauder index formula gives
\[
\text{index}_{C^0_{[0,1]}}(A_\alpha, 0) = (-1)^k \quad \text{if} \quad k^2\pi^2 < a_1 = a_2 < (k + 1)^2\pi^2.
\]
By the homotopy invariance of the degree, we see that $\text{index}(A_\alpha, 0)$ is constant in $D_k^j$ for any $j = 1, 2, k = 0, 1, 2, \cdots$. Therefore,
\[
\text{index}(A_\alpha, 0) = \begin{cases} 
1 & \text{if } (a_1, a_2) \in D_k^j, \\
-1 & \text{if } (a_1, a_2) \in D_k^j,
\end{cases} \quad (3.6)
\]
for $k = 0, 1, 2, \cdots$.

We now assume $(a_1, a_2) \in D_0^0$. Corollary 3.2 implies that $W = \{0, w_0, +\}$; so that it follows from Lemma 3.4 that
\[
1 = \text{index}(A_\alpha, 0) + \text{index}(A_\alpha, w_0, +) = \text{index}(A_\alpha, 0) + 1.
\]
Thus $\text{index}(A_\alpha, 0) = 0$ if $(a_1, a_2) \in D_0^0$. Similarly, we can get $\text{index}(A_\alpha, 0) = 0$ if $(a_1, a_2) \in D_0^1$.

We next consider the case $(a_1, a_2) \in D_1^0$. Then $W = \{0, w_0, \pm, w_1, \pm\}$ by Corollary 3.2. So Lemma 3.4 together with (3.6) leads us to
\[
1 = \text{index}(A_\alpha, 0) + \text{index}(A_\alpha, w_0, +) + \text{index}(A_\alpha, w_0, -) + \text{index}(A_\alpha, w_1, +) + \text{index}(A_\alpha, w_1, -),
\]
which implies $\text{index}(A_\alpha, w_1, +) = \text{index}(A_\alpha, w_1, -) = -1$ for any $(a_1, a_2) \in D_1^1$. Moreover, by the homotopy invariance of the degree, one can see
\[
\text{index}(A_\alpha, w_1, +) = \text{index}(A_\alpha, w_1, -) = -1
\]
for any $(a_1, a_2) \in \mathbb{R}^2$ satisfying $1/\sqrt{a_1} + 1/\sqrt{a_2} < 1/\pi$.

As the third step we study the case $(a_1, a_2) \in D_1^2$. The same reasoning as above yields
\[
1 = \text{index}(A_\alpha, 0) + \text{index}(A_\alpha, w_0, +) + \text{index}(A_\alpha, w_0, -) + \text{index}(A_\alpha, w_1, +) + \text{index}(A_\alpha, w_1, -) + \text{index}(A_\alpha, w_2, +) + \text{index}(A_\alpha, w_2, -),
\]
Hence it follows that $\text{index}(A_\alpha, w_2, +) = \text{index}(A_\alpha, w_2, -) = 1$. Moreover, one can show that $\text{index}(A_\alpha, 0) = 0$ if $(a_1, a_2) \in D_1^3 \cup D_1^4$.

In order to complete the proof, it is sufficient to repeat the preceding arguments. \(\square\)

We next consider the one-dimensional version of limit equation (1.4):
\[
\begin{cases} 
-w'' = h(w) & \text{for } 0 < x < 1, \\
w(0) = w(1) = 0,
\end{cases} \quad (3.7)
\]
where $h(w) = a_1 w^+ + w^- (a_2 + b_2 w^-)$. Nontrivial solutions of (3.7) are given by the following theorem.
Theorem 3.5. (i) If $a_1 < \pi^2$ and $a_2 \leq \pi^2$, then (3.7) admits no nontrivial solution.
(ii) If $a_1 = \pi^2$, then (3.7) has infinitely many solutions of the form $w = C \sin \pi x$ with $C \geq 0$.
(iii) If $\frac{k}{\sqrt{a_1}} + \frac{k}{\sqrt{a_2}} < \frac{1}{\pi}$ for some positive integer $k$, then (3.7) has exactly two nontrivial solutions which have exactly $2k - 1$ zero points in $(0, 1)$; one satisfies $w'(0) > 0$ and the other $w'(0) < 0$.
(iv) If $\frac{k+1}{\sqrt{a_1}} + \frac{k+1}{\sqrt{a_2}} < \frac{1}{\pi}$ (resp. $\frac{k}{\sqrt{a_1}} + \frac{k+1}{\sqrt{a_2}} < \frac{1}{\pi}$) for a positive (resp. nonnegative) integer $k$, then (3.7) has a unique nontrivial solution which has exactly $2k$ zero points in $(0, 1)$ and satisfies $w'(0) > 0$ (resp. $w'(0) < 0$).

Proof. The proof of is almost the same as that of Theorem 3.1 except for $I_1(\gamma) = \pi/\sqrt{a_1}$ for all $\gamma$, where $I_1$ is a time-map for the initial value problem associated with (3.7).

Remark 3.3. In Theorem 3.5 we should note that there exist no positive solutions of (3.7) if $a_1 \neq \pi^2$.

We will calculate the index of each solution of (3.7) as in Theorem 3.3.

Theorem 3.6. (i) Each nontrivial solution $w_{k, \pm}$, $k = 1, 2, 3, \ldots$, of (3.7) satisfies
\[ \text{index}_{S_{[0,1]}(A_{\infty}, w_{k, \pm})} = (-1)^k. \]  

(ii) The trivial solution of (3.7) satisfies
\[ \text{index}_{S_{[0,1]}(A_{\infty}, 0)} = \begin{cases} 1 & \text{if } (a_1, a_2) \in D_k^1, \\
-1 & \text{if } (a_1, a_2) \in D_k^2, \\
0 & \text{if } (a_1, a_2) \in D_k^3 \cup D_k^4, \end{cases} \]  

where $A_{\infty}$ is defined as in Theorem 2.3.

The proof of Theorem 3.6 is quite similar to Theorem 3.3 and is based on the following lemma which corresponds to Lemma 3.4:

Lemma 3.7. (i) If $R$ is sufficiently large, then it holds that
\[ \text{deg}_{S_{[0,1]}(I - A_{\infty}, B_R, 0)} = \begin{cases} 0 & \text{if } a_1 > \lambda_1, \\
1 & \text{if } a_1 < \lambda_1. \end{cases} \]

(ii) For every $a_2 > \lambda_1$ a unique negative solution $w_{0,-}$ of (3.7) satisfies
\[ \text{index}_{S_{[0,1]}(A_{\infty}, w_{0,-})} = 1. \]

Proof. For the proof see Dancer and Du [6, pp.471–473].
Denote by $S_\alpha$ (resp. $S_\infty$) the set of changing sign solutions of (3.1) (resp. (3.7)). We have shown that $n(S_\alpha) :=$ the number of solutions in $S_\alpha$ is given by

$$n(S_\alpha) = \begin{cases} 4k - 2 & \text{if } (a_1, a_2) \in D_1^k, \\ 4k & \text{if } (a_1, a_2) \in D_2^k, \\ 4k - 1 & \text{if } (a_1, a_2) \in D_3^k \cup D_4^k, \end{cases}$$

and that every changing sign solution is isolated and has non-zero index by Theorem 3.3. The same results are also valid for $S_\infty$. Therefore, by Theorems 2.2 and 2.3, one can deduce the following multiplicity result for positive solutions of (1.1):

**Theorem 3.8.** Let $a_1 > \pi^2$ and $a_2 > \pi^2$. Then there exist positive constants $\bar{b}_1, \bar{b}_2$ such that for every $b_1 > \bar{b}_1$, $b_2 > \bar{b}_2$,

(a) there exist at least $4k - 2$ positive solutions of (1.1) if $(a_1, a_2) \in D_1^k$,

(b) there exist at least $4k$ positive solutions of (1.1) if $(a_1, a_2) \in D_2^k$,

(c) there exist at least $4k - 1$ positive solutions of (1.1) if $(a_1, a_2) \in D_3^k \cup D_4^k$.

**Remark 3.4.** Let $\{b_{1,n}, b_{2,n}\}$ be a sequence satisfying $b_{1,n} > \bar{b}_1, b_{2,n} > \bar{b}_2$ ($n = 1, 2, \cdots$) and

$$b_{1,n}, b_{2,n} \to \infty \quad \text{and} \quad \frac{b_{2,n}}{b_{1,n}} \to \alpha \in (0, \infty] \quad \text{as } n \to \infty.$$

Let $(u_n, v_n)$ be any positive solution of (1.1) with $b_1 = b_{1,n}$ and $b_2 = b_{2,n}$. Then combining Theorem 3.8 and the results of Dancer and Du [5] one can choose a subsequence $\{(u_{n'}, v_{n'})\}$ of $\{(u_n, v_n)\}$ such that either

(i) $(u_{n'}, v_{n'}) \to ((w_k)^+/\alpha, -(w_k)^{-})$ in $C[0, 1] \times C[0, 1]$ for some positive integer $k$

or

(ii) $(u_{n'}, v_{n'}) \to (0, 0)$ in $C[0, 1] \times C[0, 1]$.

We will give some numerical examples. Let $a_1 = 30$, $a_2 = 60$, $b_1 = b_2 = 1$ and $\alpha = 1$. It is easy to see $(a_1, a_2) \in D_1^1$; so that Theorem 3.8 tells us that (3.1) has two changing sign solutions $w_{1,\pm}$ with a unique zero-point in $(0, 1)$. Figure 2 shows the profile of $w_{1,\pm}$. Consider $(u, v)$ with $u = (w_{1,\pm})^+$ and $v = -(w_{1,\pm})^-$ as in Figure 3; this pair is regarded as a limit solution of (1.1) as $b_1, b_2 \to \infty$ with $b_2/b_1 \to 1$. Figure 4 exhibits positive solutions of (1.1) corresponding to $b_1 = b_2 = 50, 100, 200, 500, 1000, 10000$.

### 4 Profiles of Positive Solutions

Numerical examples in the preceding section suggest that profiles of positive solutions of (1.1) for large $b_1, b_2$ are quite similar to those of the corresponding limit solutions. In this section we will study the profiles of positive solutions of (1.1) in detail.

**Theorem 4.1.** Let $(u, v)$ be a positive solution of (1.1). Denote by $M_u$ (resp. $M_v$) the set of local maximum points of $u$ (resp. $v$) in $(0, 1)$. Then

$$n(M_u) - n(M_v) = 0 \; \text{or} \; \pm 1,$$
where \( n(M_u) \) (resp. \( n(M_v) \)) denotes the number of elements in \( M_u \) (resp. \( M_v \)). Moreover, if \( x_0, x_1 \in M_u \) with \( x_0 < x_1 \) are adjacent points in \( M_u \), then there exists a unique point \( y_0 \in M_u \) between \( x_0 \) and \( x_1 \). That is, local maximum points of \( u \) and \( v \) appear alternately.

To prove this theorem, we follow the arguments used by Nakashima [13], where it is shown that the number of zero points of \( u' \) and \( v' \) are identical for a certain class of competition diffusion systems with Neumann boundary conditions.

Before giving the proof of Theorem 4.1 we prepare a series of lemmas.

**Lemma 4.2.** Let \( (u,v) \) be a positive solution of (1.1). If \( v' \geq 0 \) (resp. \( v' \leq 0 \)) for \( x_0 < x < x_1 \), \( u'(x_0) \leq 0 \) and \( u'(x_1) \leq 0 \) (resp. \( u'(x_0) \geq 0 \) and \( u'(x_1) \geq 0 \)), then \( u' < 0 \) (resp. \( u' > 0 \)) for \( x_0 < x < x_1 \).

**Proof.** Denote by \( \sigma_1(q; z_1, z_2) \) (with \( z_1 < z_2 \)) the first eigenvalue of \(-u'' + qw\) in \((z_1, z_2)\) with \( w(z_1) = w(z_2) = 0 \). Since \( u \) is a positive function satisfying the first equation of (1.1), one can see

\[
\sigma_1(u + b_1 v - a_1; 0, 1) = 0.
\]

If we set \( w = u' \), then \( w \) satisfies

\[
-w'' + (2u + b_1 v - a_1)w = -b_1 w u'.
\]  \hspace{1cm} (4.1)

Here observe that

\[
\sigma_1(2u + b_1 v - a_1; 0, 1) > \sigma_1(u + b_1 v - a_1; 0, 1) = 0.
\]

Therefore, taking account of the variational characterization of the first eigenvalue one can deduce

\[
\sigma_1(2u + b_1 v - a_1; x_0, x_1) > 0 \quad \text{for any} \quad 0 \leq x_0 < x_1 \leq 1;
\]

so that we can apply the strong maximum principle to (4.1) (see, e.g., [14, Lemma 7]). By \( v'(x) \geq 0 \) for \( x_0 < x < x_1 \), the right-hand side of (4.1) is non-positive and \( u'(x_0) \leq 0, u'(x_1) \leq 0 \). Hence it follows from the strong maximum principle that \( u'(x) < 0 \) for \( x_0 < x < x_1 \). \hfill \Box

**Lemma 4.3.** If \( u' > 0 \) (resp. \( u' < 0 \)) for \( x_0 < x < x_1 \) and \( u'(x_0) = u'(x_1) = 0 \), then there exists a point \( \bar{x} \in (x_0, x_1) \) satisfying \( u'(<\bar{x}) < 0 \) (resp. \( u'(<\bar{x}) > 0 \)).

**Proof.** Suppose \( v' \geq 0 \) for \( x_0 < x < x_1 \). Then Lemma 4.2 yields \( u' < 0 \) for \( x_0 < x < x_1 \), which is a contradiction. \hfill \Box

Making use of Lemma 4.3 we follow the proof of Nakashima [13, Proposition 3.1] to show the following lemma.

**Lemma 4.4.** The number of zero points of \( u' \) and that of \( v' \) are finite.

Moreover, one can obtain the following lemma from Lemmas 4.3 and 4.4 in the same way as [13, Proposition 3.2].
Lemma 4.5. The sign of \( u' \) (resp. \( v' \)) changes at every zero point of \( u' \) (resp. \( v' \)).

We are ready to prove Theorem 4.1.

Proof of Theorem 4.1. We will prove this theorem with use of the idea developed in [13, Theorem 2.3]. By Lemma 4.4, the number of local maximum points of \( u' \) (resp. \( v' \)) is finite. Set
\[ n = n(M_a) \quad \text{and} \quad m = n(M_v). \]
In view of \( u'(0) > 0 \), \( u'(1) < 0 \) and Lemma 4.5, we can easily show that \( u' \) has \( 2n - 1 \) zero points in \((0, 1)\); denote these zero points of \( u' \) by
\[ x_1 < x_2 < x_3 < \cdots < x_{2n-1}. \]
Obviously, \( M_a = \{ x_1, x_3, x_5, \ldots, x_{2n-1} \} \) and
\[ u' < 0 \quad \text{for} \quad x_{2k-1} < x < x_{2k} \quad \text{and} \quad u' > 0 \quad \text{for} \quad x_{2k} < x < x_{2k+1} \]
with \( k = 1, 2, \ldots, n-1 \). Lemma 4.3 implies that there exist \( y_i \in (x_i, x_{i+1}), i = 1, 2, \ldots, 2n - 2 \), such that
\[ v'(y_{2k-1}) > 0 \quad \text{and} \quad v'(y_{2k}) < 0, \quad \text{with} \quad k = 1, 2, \ldots, n-1. \]
Hence \( v' \) has at least \( 2n - 3 \) zero points in \((0, 1)\) and, therefore, it has at least \( n - 1 \) local maximum points in \((0, 1)\). Thus we have shown \( m \geq n - 1 \). Similarly, \( n \geq m - 1 \); so that
\[ |n - m| \leq 1. \]
We next prove the latter half. Suppose that \( x_0, x_1 \in M_a \) with \( x_0 < x_1 \) and that \( u \) has no local maximum in \((x_0, x_1)\). Then it follows from Lemma 4.5 that \( u \) has a unique local minimum at \( x = x_2 \in (x_0, x_1) \); so that
\[ u' < 0 \quad \text{for} \quad x_0 < x < x_2 \quad \text{and} \quad u' > 0 \quad \text{for} \quad x_2 < x < x_1. \]
By Lemma 4.3, there exist \( y_0 < y_1 \) such that
\[ v'(y_0) > 0 \quad \text{and} \quad v'(y_1) < 0 \quad \text{with} \quad x_0 < y_0 < x_2 < y_1 < x_1. \]
Hence there exists at least one point \( z_0 \in M_v \cap (y_0, y_1) \). We will prove that \( z_0 \) is a unique local maximum point of \( v \) in \((x_0, x_1)\). Assume that \( z_0 < z_1 \) are two local maximum points of \( v \) in \((x_0, x_1)\). Repeating the previous argument one can show that \( u \) has at least one local maximum point in \((z_0, z_1)\), which is a contradiction. Thus the proof is complete. \( \square \)

We study the case when \( b_{1,n} \) and \( b_{2,n} \) (\( n = 1, 2, 3, \ldots \)) satisfy \( b_{1,n} > \bar{b}_1, b_{2,n} > \bar{b}_2 \) and
\[ b_{1,n}, b_{2,n} \to \infty \quad \text{and} \quad \frac{b_{2,n}}{b_{1,n}} \to \alpha \in (0, \infty) \quad \text{as} \quad n \to \infty. \]
Here \( \bar{b}_1 \) and \( \bar{b}_2 \) are positive constants appearing Theorem 3.8. Consider the following problem:
\[
\begin{cases}
-u'' = u(a_1 - u - b_{1,n}v) & \text{for } 0 < x < 1, \\
-v'' = v(a_2 - v - b_{2,n}u) & \\
u(0) = u(1) = v(0) = v(1) = 0,
\end{cases}
\]  
(4.2)
and let \((u_n, v_n)\) be a positive solution of (4.2) such that \(\{(u_n, v_n)\}\) is a convergent sequence in \(L^2(0, 1) \times L^2(0, 1)\) (see [5]); so

\[
u_n \to \frac{w^+}{\alpha} \quad \text{and} \quad v_n \to -w^- \quad \text{in} \quad L^2(0, 1) \quad (4.3)
\]

for some \(w\) which is a changing sign solution of (3.1). Taking \(L^2\)-inner products of equations in (4.2) with \(u_n\) and \(v_n\) leads to

\[
\|u'_n\|_2^2 \leq a_1 \|u_n\|_2^2 \leq a_1^3 \quad \text{and} \quad \|v'_n\|_2^2 \leq a_2 \|v_n\|_2^2 \leq a_2^3.
\]

If we use Gagliardo-Nirenberg's inequality, we see

\[
\begin{align*}
\|u_m - u_n\|_\infty & \leq C \|u_m - u_n\|_2^{1/2} \|u'_m - u'_n\|_2^{1/2}, \\
\|v_m - v_n\|_\infty & \leq C \|v_m - v_n\|_2^{1/2} \|v'_m - v'_n\|_2^{1/2}
\end{align*}
\]

for \(m > n\), with some \(C > 0\). Hence one can get stronger convergence results than (4.3);

\[
\lim_{n \to \infty} u_n = \frac{w^+}{\alpha} \quad \text{and} \quad \lim_{n \to \infty} v_n = -w^- \quad \text{uniformly in} \quad [0, 1]. \quad (4.4)
\]

If we put

\[
w_n := \frac{b_{2,n}}{b_{1,n}} u_n - v_n,
\]

then (4.4) implies that

\[
w_n \to w^+ + w^- = w \quad \text{in} \quad C[0, 1]. \quad (4.5)
\]

Clearly, \(w_n\) satisfies

\[-w''_n = \frac{b_{2,n}}{b_{1,n}} (a_1 - u_n) - v_n(a_2 - v_n),
\]

whose right-hand side converges as \(n \to \infty\) to

\[w^+(a_1 - w^+/\alpha) + w^-(a_2 + w^-)
\]

in \(C[0, 1]\) by (4.4). Since \(w\) satisfies (3.1), the above convergence means

\[
w''_n \to w'' \quad \text{in} \quad C[0, 1]. \quad (4.6)
\]

In view of \(\|w'\|_\infty \leq \|w''\|_1\), for any \(w \in C^2[0, 1]\) such that \(w(0) = w(1) = 0\), we see from (4.6) that

\[w'_n \to w' \quad \text{in} \quad C[0, 1]. \quad (4.7)
\]

Thus we have shown

**Lemma 4.6.** Assume that \(\{u_n, v_n\}\) satisfies (4.3). Then \(\{w_n\}\) satisfies

\[w_n \to w \quad \text{in} \quad C^2[0, 1] \quad \text{as} \quad n \to \infty.
\]
More information can be obtained by the following theorem.

**Theorem 4.7.** (i) If \((u_n, v_n)\) satisfies
\[
(u_n, v_n) \to \left( (w_{2k-1,-}^+)^\ast_\ast /\alpha, -(w_{2k-1,+}^-)^* \right) \quad \text{in} \quad L^2(0, 1) \times L^2(0, 1) \quad \text{as} \ n \to \infty,
\]
then for any \(\varepsilon > 0\) there exists a positive integer \(n_0\) such that for any \(n \geq n_0\)
(a) \(u_n\) has precisely \(k\) local maximum points in \((0, 1 - \varepsilon)\),
(b) \(v_n\) has precisely \(k\) local maximum points in \((\varepsilon, 1)\).

(ii) If \((u_n, v_n) \to \left( (w_{2k-1,-}^+)^\ast_\ast /\alpha, -(w_{2k-1,+}^-)^* \right)\) in \(L^2(0, 1) \times L^2(0, 1)\) as \(n \to \infty\), then for any \(\varepsilon > 0\) there exists a positive integer \(n_0\) such that for any \(n \geq n_0\)
(a) \(u_n\) has precisely \(k\) local maximum points in \((\varepsilon, 1)\),
(b) \(v_n\) has precisely \(k\) local maximum points in \((0, 1 - \varepsilon)\).

(iii) If \((u_n, v_n) \to \left( (w_{2k,+}^+)^\ast_\ast /\alpha, -(w_{2k,+}^-)^* \right)\) in \(L^2(0, 1) \times L^2(0, 1)\) as \(n \to \infty\), then for any \(\varepsilon > 0\) there exists a positive integer \(n_0\) such that for any \(n \geq n_0\)
(a) \(u_n\) has precisely \(k + 1\) local maximum points in \((0, 1)\),
(b) \(v_n\) has precisely \(k\) local maximum points in \((\varepsilon, 1 - \varepsilon)\).

(iv) If \((u_n, v_n) \to \left( (w_{2k,+}^+)^\ast_\ast /\alpha, -(w_{2k,+}^-)^* \right)\) in \(L^2(0, 1) \times L^2(0, 1)\) as \(n \to \infty\), then for any \(\varepsilon > 0\) there exists a positive integer \(n_0\) such that for any \(n \geq n_0\)
(a) \(u_n\) has precisely \(k\) local maximum points in \((\varepsilon, 1 - \varepsilon)\),
(b) \(v_n\) has precisely \(k + 1\) local maximum points in \((0, 1)\).

**Proof.** We only prove (i) for \(k = 1\). The other cases can be treated in a similar manner.

Suppose that \((u_n, v_n) \to \left( (w_{1,+}^+)^\ast_\ast /\alpha, -(w_{1,+}^-)^* \right)\) as \(n \to \infty\) in \(L^2(0, 1) \times L^2(0, 1)\). By Lemma 4.6,
\[
w_n \to w_{1,+}^\ast \quad \text{in} \quad C^2[0, 1]. \tag{4.8}
\]

Let \(\overline{x}\) be a zero point of \(w_{1,+}^\ast\) in \((0, 1)\). We will prove that \(u_n\) has no local maximum in \((\overline{x}, 1 - \varepsilon)\) for sufficiently large \(n\). Suppose that \(u_n\) has a local maximum point \(x_n\) in \((\overline{x}, 1)\). Since
\[
-w_n'(x_n) = u_n(x_n) \left( a_1 - u_n(x_n) - b_{1,n}v_n(x_n) \right) \geq 0,
\]
we obtain
\[
v_n(x_n) \leq \frac{1}{b_{1,n}} \left( a_1 - u_n(x_n) \right) \to 0 \quad \text{as} \ n \to \infty. \tag{4.9}
\]

If \(\{x_n\}\) has a subsequence \(\{x_n'\}\) such that \(\lim_{n' \to \infty} x_n' = x^* \in (\overline{x}, 1)\), it follows from (4.4) that
\[
\lim_{n' \to \infty} v_n'(x_n') = -(w_{1,+}^-)^*(x^*) > 0,
\]
which is a contradiction to (4.9). Therefore, it holds that either \(x_n \to \overline{x}\) or \(x_n \to 1\).

Assume \(x_n \to \overline{x}\). Let \(\widehat{x}_n\) be a local minimum point of \(u_n\), which exists on the left-hand side of \(x_n\). Then \(\widehat{x}_n \to \overline{x}\) and
\[
u_n'(x_n) = u_n'(x_n) > 0 \quad \text{for} \quad \widehat{x}_n < x < x_n \quad \text{and} \quad u_n'(x_n) = u_n'(\widehat{x}_n) = 0.
\]

Lemma 4.3 implies that there exists \(y_n \in (\widehat{x}_n, x_n)\) such that \(v_n'(y_n) < 0\). Moreover, it follows from (4.8) that
\[
u_n'(x_n) = -v_n'(x_n) \to (w_{1,+}^\ast)'(\overline{x}) < 0 \quad \text{as} \ n \to \infty;
\]
which assures \( v'_n(x_n) > 0 \) for sufficiently large \( n \). Hence there exists \( z_n \in (y_n, x_n) \) such that \( v'(z_n) = 0 \). Since \( z_n \to \bar{x} \) and

\[
    u'_n(z_n) = \frac{b_{2,n}}{b_{1,n}} u'_n(z_n) > 0,
\]

we see from (4.8) that \( (w_{1,\tau})'(\bar{x}) \geq 0 \), which is a contradiction to \( (w_{1,\tau})'(\bar{x}) < 0 \). Thus we have shown \( x_n \to 1 \). This fact means that \( u_n \) has no local maximum in \((\bar{x}, 1-\varepsilon)\) for sufficiently large \( n \).

Similarly, we can prove that \( v_n \) has no local maximum point in \((\varepsilon, \bar{x})\) for sufficiently large \( n \).

We will also prove that \( u_n \) has precisely one local maximum point in \((0, \bar{x})\). Suppose that \( u_n \) has two local maximum points \( x_n < \bar{x}_n < \bar{x}_n \) in \((0, \bar{x})\). Let \( \hat{x}_n \in (x_n, \bar{x}_n) \) be a local minimum point of \( u_n \). Then

\[
    u'_n < 0 \text{ for } x_n < x < \hat{x}_n \quad \text{and} \quad u'_n > 0 \text{ for } \hat{x}_n < x < \bar{x}_n.
\]

By Theorem 4.1, \( v_n \) has precisely one local maximum at \( y_n \in (x_n, \bar{x}_n) \). The same reasoning as in the previous step leads us to derive \( y_n \to 0 \) and, therefore, \( x_n \to 0 \). By (4.7)

\[
    u'_n(x_n) = -v'_n(x_n) \quad \to \quad (w_{1,\tau})'(0) > 0,
\]

we see \( v'_n(x_n) < 0 \) for sufficiently large \( n \). Therefore, \( v_n \) has a local minimum at \( \hat{y}_n \in (x_n, y_n) \). Since

\[
    u'_n(\hat{y}_n) = \frac{b_{2,n}}{b_{1,n}} u'_n(\hat{y}_n) \quad \to \quad (w_{1,\tau})'(0) > 0
\]

from (4.7), one can get \( u'_n(\hat{y}_n) > 0 \) for sufficiently large \( n \). Thus it is seen that \( x_n < \hat{x}_n < \hat{y}_n < y_n < \bar{x}_n \), and, therefore,

\[
    u'_n(x) > 0 \quad v'_n(x) \geq 0 \quad \text{for } \hat{y}_n < x < y_n \quad \text{and} \quad v'_n(\hat{y}_n) = v'_n(y_n) = 0.
\]

This is a contradiction to Lemma 4.3; so that we have shown that \( u_n \) has a unique local maximum in \((0, \bar{x})\).

Similarly, it is also possible to show that \( v_n \) has precisely one local maximum in \((\varepsilon, 1)\). \hfill \Box

## 5 Some Remarks on Nonstationary Problems

In this section we will briefly discuss nonstationary problems associated with (1.1) and (3.1) (or (3.7)).

Consider the following parabolic system associated with (1.1):

\[
\begin{align*}
    u_t - u_{xx} &= u(a_1 - u - b_1v) & \text{in} \quad (0, 1) \times (0, \infty), \\
    v_t - v_{xx} &= v(a_2 - v - b_2u) & \text{in} \quad (0, 1) \times (0, \infty), \\
    u(0, t) &= u(1, t) = v(0, t) = v(1, t) = 0 & \text{in} \quad (0, \infty), \\
    u(\cdot, 0) &= u_0, \quad v(\cdot, 0) = v_0 & \text{in} \quad (0, 1),
\end{align*}
\]

(5.1)
where \( u_0 \) and \( v_0 \) are nonnegative functions of class \( C[0, 1] \). Clearly, (5.1) admits a unique global solution \((u, v)\) such that

\[
u, v \in C([0, 1] \times [0, \infty)) \cap C^{2,1}([0, 1] \times [\delta, \infty)) \quad \text{for any } \delta > 0.\]

Assume that \( b_{1,n}, b_{2,n} > 0 \) and \( u_{0,n}, v_{0,n} \in C[0, 1] \) \((n = 1, 2, \ldots)\) satisfy

\[
\begin{align*}
b_{1,n}, b_{2,n} & \to \infty, & \frac{b_{2,n}}{b_{1,n}} & \to \alpha \in (0, \infty], \\
\frac{b_{2,n}}{b_{1,n}} u_{0,n} - v_{0,n} & \to u_0 \quad \text{weakly in } L^2(0,1).
\end{align*}
\]

Denote by \((u_n, v_n)\) the solution of (5.1) with \( b_1 = b_{1,n}, b_2 = b_{2,n}, u_0 = u_{0,n} \) and \( v_0 = v_{0,n} \).

**Theorem 5.1.** (i) If \( \alpha < \infty \), then for any \( T > 0 \) there exists a subsequence \( \{(u_{n'}, v_{n'})\} \) of \( \{(u_n, v_n)\} \) such that

\[
u_{n'} \to \frac{w^+}{\alpha} \quad \text{and} \quad v_{n'} \to -w^- \quad \text{in } L^1((0,1) \times (0, T)),
\]

where \( w \) is a weak solution of

\[
\begin{align*}
w_t - w_{xx} &= w^+ (a_1 - w^+ / \alpha) + w^- (a_2 + w^-) \quad \text{in } (0,1) \times (0, \infty), \\
w(0, t) &= w(1, t) = 0 \\
w(\cdot, 0) &= w_0 \quad \text{in } (0,1),
\end{align*}
\]

(ii) If \( \alpha = \infty \), then for any \( T > 0 \) there exists a subsequence \( \{(u_{n'}, v_{n'})\} \) of \( \{(u_n, v_n)\} \) such that

\[
\frac{b_{2,n}}{b_{1,n}} u_{n'} \to \frac{w^+}{\alpha'} \quad \text{and} \quad v_{n'} \to -w^- \quad \text{in } L^1((0,1) \times (0, T)),
\]

where \( w \) is a weak solution of

\[
\begin{align*}
w_t - w_{xx} &= a_1 w^+ + w^- (a_2 + w^-) \quad \text{in } (0,1) \times (0, \infty), \\
w(0, t) &= w(1, t) = 0 \\
w(\cdot, 0) &= w_0 \quad \text{in } (0,1).
\end{align*}
\]

The proof of Theorem 5.1 is similar to Dancer et al. [9, Lemma 3.1–3.5]. So we omit it here.

Let \( w^\ast \) be any changing sign solution of (3.1) or (3.7); so that \( w^\ast \) is regarded as a stationary solution of (5.2) or (5.3). From Remark 2.2, if \( w^\ast \) is unstable, then a positive solution \((u, v)\) of (1.1) which is near to \((w^\ast)^+ / \alpha, -(w^\ast)^-\) is unstable when \( b_1, b_2 \) are sufficiently large and \( b_2/b_1 \) is close to \( \alpha \).

We will show the instability of \( w^\ast \). This fact implies that all positive solutions of (1.1) constructed in Theorem 3.8 are unstable for sufficiently large \( b_1, b_2 \).

**Proposition 5.2.** Any changing sign solution of (3.1) (3.7) is unstable.
Proof. We only prove the instability of $w_{1,+}$ of (3.1) because the other solutions can be discussed essentially in the same way. We use the same notation as in Section 3 and let $w(x, \gamma)$ be the solution of (3.2). It follows from the proofs of Theorems 3.1 and 3.5 that $w_{1,+} = w(\cdot, \overline{\gamma})$ for some $\overline{\gamma} > 0$. One can show that, for any small $\varepsilon > 0$, $w(\cdot, \overline{\gamma} + \varepsilon)$ is a lower solution of (3.1) and that $w(\cdot, \overline{\gamma} - \varepsilon)$ is an upper solution. If we take $w_0 = w(\cdot, \overline{\gamma} + \varepsilon)$ in (5.2), then the monotone argument enables us to conclude that the corresponding solution of (5.1) converges to a unique positive solution $w_{0,+}$ as $t \to +\infty$. Similarly, if we take $w_0 = w(\cdot, \overline{\gamma} - \varepsilon)$ in (5.2), then the solution of (5.2) converges to a unique negative solution $w_{0,-}$. These results yield the instability of $w_{1,+}$. □

References


Takefumi Hirose and Yoshio Yamada  
Department of Mathematical Sciences,  
Waseda University  
3-4-1 Ohkubo, Shinjuku-ku, 169-8555 Tokyo  
JAPAN

![Figure 1: Domain $D_k^j$](image)
Figure 2: Profile of $w_{1,}$

Figure 3: Limit solution

$b_1 = b_2 = 50$

$b_1 = b_2 = 100$

$b_1 = b_2 = 200$

$b_1 = b_2 = 500$

$b_1 = b_2 = 1000$

$b_1 = b_2 = 10000$

Figure 4: Positive solutions of (1.1) with $a_1 = 30, a_2 = 60$