Quantum cohomology: is it still relevant?

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SOME HISTORY

First we sketch the general ideas (and some sociology) of quantum cohomology. Technical material is postponed to the following sections.
In quantum theory it is necessary to consider integrals over “the space of all paths”, but a rigorous mathematical formulation of such integrals (over infinite-dimensional spaces) is a difficult problem.

Physicists were led to consider the space

$$\text{Hol}(\Sigma, M) = \{ f : \Sigma \to M \mid f \text{ is holomorphic} \}$$

where

- $\Sigma$ = compact Riemann surface
- $M$ = compact Kähler manifold

The connected components of this space are generally noncompact, and may have singularities, but at least they are finite-dimensional.
For example, holomorphic maps from the Riemann sphere $\Sigma = \mathbb{CP}^1$ to $M = \mathbb{CP}^n$ are given by $(n + 1)$-tuples of polynomials

$$[p_0(z); \ldots; p_n(z)]$$

and the connected components of $\text{Hol}(\Sigma, M)$ are indexed by $d = 0, 1, 2, \ldots$

where $d = \max\{\deg p_0, \ldots, \deg p_n\}$ (and $p_0, \ldots, p_n$ have no common factor).

It is plausible that a theory of integration over such spaces might exist.

PERSONAL ASIDE: Algebraic topologists studied the homotopy groups and homology groups of such spaces. 1970’s: G. Segal, F. Cohen,...; 1990’s: M. Guest - A. Kozlowski - K. Yamaguchi,...
In fact, a mathematical foundation for such a theory of integration — more generally, for pseudo-holomorphic curves in symplectic manifolds — was constructed by Mikhail Gromov in the 1980’s. This involved a deep study of the (non)compactness of the space. Algebraic geometers provided another approach: in the 1990’s Maxim Kontsevich introduced the concept of stable curve, in order to construct an “improved” version of $\text{Hol}(\Sigma, M)$. Both approaches — moduli spaces of curves, in symplectic geometry and algebraic geometry — became major areas of research.
Development of quantum cohomology

(Phase I, Phase II, Phase III)
Phase I: Gromov-Witten invariants

Quantum cohomology was born (into the mathematical world) in the 1990’s as “intersection theory” on such moduli spaces.

When the holomorphic maps are constant, quantum cohomology reduces to intersection theory on the target manifold $M$ itself, i.e. ordinary cohomology. Thus, quantum cohomology appeared to be a rather natural generalization of ordinary cohomology.
On the other hand, quantum cohomology is very different from ordinary cohomology in several respects. First, it is not (in any naive sense) functorial. This means that quantum cohomology is difficult to compute. Second, it does not (directly) measure any topological quantity, as the symplectic/complex structure plays an essential role.

In view of the lack of functoriality, early research on quantum cohomology concentrated on “Gromov-Witten invariants”. These are the raw data of intersection theory.

They amount to giving a list of all possible intersections.
Phase II: quantum differential equations

In the next period of development, these Gromov-Witten invariants were organised more systematically. This involved something quite new: a relation with differential equations.

The famous predictions of Mirror Symmetry — counting rational curves in Calabi-Yau manifolds — came from solving differential equations. This is explained in a 1991 paper by Philip Candelas, Xenia de la Ossa, Paul Green, and Linda Parkes.

(We are ordering the “Phases” in logical order, not time order.)
We take the starting point of Phase II as the address by Alexander Givental at the 1994 ICM in Zürich.

Givental introduced the “quantum differential equations”, a linear system of partial differential equations.

The key property of the quantum differential equations is that there exists a solution whose series expansion is a generating function of the (generalized) Gromov-Witten invariants.
The dependent and independent variables of the quantum differential equations were coordinates on cohomology vector spaces. This means cohomology with complex coefficients; the coefficients of quantum differential equations were holomorphic. To a topologist, doing calculus on cohomology spaces is unusual, to say the least. Givental called this new phenomenon “homological geometry”.

It was a major achievement to give a mathematical treatment of the physicists’ original Mirror Symmetry predictions for (certain) Calabi-Yau manifolds. This was done by Givental and also by Bong Lian, Kefeng Liu, and Shing-Tung Yau.
Before going on to Phase III, it should be mentioned that the symplectic approach saw its own dramatic developments. The key words here were Floer theory and Witten’s version of Morse theory.

Floer theory for the loop space $\Lambda M$ is closely related to the quantum cohomology of $M$, but Floer theory developed in its own way. In particular the case of a target manifold $M$ with Lagrangian submanifold $L$, and Gromov-Witten invariants based on maps from $(D, \partial D)$ to $(M, L)$, became a major theme, developed by Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. Here $D$ is a two-dimensional disk with boundary $\partial D$. 
Phase III: integrable systems

In this narrative, Phase III started with another ICM talk, namely that of Boris Dubrovin in 1998. Here another mathematical leap forward was made: from linear to nonlinear differential equations.
As always in this story, physicists had anticipated such a leap, with their WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equations and tt* (topological-antitopological fusion) equations. These equations involve more general kinds of quantum cohomology:

— WDVV describes “big” quantum cohomology
— tt* describes “quantum cohomology with a real structure”.

Both are complicated and difficult to study directly, partly because of the physicists’ preference for coordinates and tensor notation. Dubrovin introduced a more intrinsic point of view for both, based on his concept of Frobenius manifold.
In the context of quantum cohomology, the Frobenius manifold is the cohomology vector space of the target manifold, endowed with further differential geometric data.

(After all, having started to do calculus on cohomology vector spaces, why not introduce connections and curvature as well?)

Independently from quantum cohomology, this kind of structure had already been investigated by Kyoji Saito in the 1980’s, in the context of singularity theory. Thus Frobenius manifolds linked quantum cohomology with singularity theory in a systematic way.
The stage was set for another unexpected application/idea of quantum cohomology: the existence of a target space $M$ (or rather, its quantum cohomology) implies the existence of a solution of a highly nontrivial nonlinear p.d.e.

For example, Dubrovin showed that the WDVV equation for $\mathbb{C}P^2$ reduces to a case of the Painlevé VI equation, so the “big” quantum cohomology of $\mathbb{C}P^2$ corresponds to a certain solution of this equation.

Dubrovin also began the study of integrable hierarchies generated by the quantum differential equations, in analogy with the KdV hierarchy which is generated by the Schrödinger equation.
In this section we give some examples related to Phase II and Phase III, to prepare some terminology for the next section. We use only undergraduate-level material in this section.
The simplest version of the quantum differential equation for $\Sigma = \mathbb{C}P^1$, $M = \mathbb{C}P^n$ is

$$(\hbar \partial)^{n+1} y = q y$$

where

- $q$ is a complex variable
- $y = y(q)$
- $\partial = q \frac{d}{dq}$
- $\hbar$ is a parameter
This is related to the cohomology of $\mathbb{CP}^n$ in the following way.

The cohomology vector space $H^*(\mathbb{CP}^n; \mathbb{C})$ has $1, b, b^2, \ldots, b^n$ as a basis, where $b$ is an (additive) generator of $H^2(\mathbb{CP}^n; \mathbb{C})$. It is generated multiplicatively by $b$, with the relation $b^{n+1} = 0$.

The quantum cohomology is equal to $H^*(\mathbb{CP}^n; \mathbb{C})$ as a vector space, but it has a “deformed” product operation, leading to the relation $b^{n+1} = q$.

(Ordinary cohomology is recovered from quantum cohomology by setting $q = 0$.)

The relation $b^{n+1} - q$ corresponds to the operator $(h\partial)^{n+1} - q$ in an obvious way.
Although $q = 0$ is a singular point for the quantum differential equation, a basis $y^{[0]}, \ldots, y^{[n]}$ of solutions (near $q = 0$) may easily be found by the Frobenius Method.

Givental observed that, if we write

$$J = (y^{[0]}, \ldots, y^{[n]}) \longleftrightarrow y^{[0]} 1 + y^{[1]} b + \cdots + y^{[n]} b^n,$$

then this Frobenius solution is given by the attractive formula

$$J(q) = q^{b/h} \sum_{k=0}^{\infty} \frac{q^k}{(b + h)(b + 2h) \cdots (b + k\hbar)}.$$

At first this looks strange, as $b$ is a cohomology class, but it makes sense as $b^{n+1} = 0$ (in cohomology). With this understanding, it is easy to verify that $(\hbar \partial)^{n+1} J = qJ$, as asserted.

\[ i.e. \ (1 + b)^{-1} = 1 - b + b^2 - \cdots \ etc. \]
There are good reasons to identify

\[ q \leftrightarrow q b \in H^2(\mathbb{C}P^n; \mathbb{C}), \]

so we have a map

\[ J : H^2(\mathbb{C}P^n; \mathbb{C}) \rightarrow H^*(\mathbb{C}P^n; \mathbb{C}) \]

(strictly speaking, a multivalued map, as \( q^{b/\hbar} = e^{(b/\hbar) \log q} \) gives rise to logarithms).

The solution of the quantum differential equation is a cohomology-valued function on a cohomology vector space!
A less obvious example is the case of a (smooth) hypersurface $M$ of degree 3 in $\mathbb{C}P^4$. There is a generator $b \in H^2(M; \mathbb{C}) \cong \mathbb{C}$ which satisfies $b^4 = 27q b^2$, but the quantum differential equation is

$$((\hbar \partial)^4 - 27q(\hbar \partial)^2 - 27\hbar q(\hbar \partial) - 6\hbar^2 q) \ y = 0.$$ 

Replacing $\hbar \partial$ by $b$ in the differential operator gives $b^4 - 27qb^2 - 27\hbar qb - 6\hbar^2 q$, and this gives $b^4 - 27qb^2$ after putting $\hbar = 0$. Here the inverse procedure is not immediately visible. However, it turns out that there is such a procedure, and here the parameter $\hbar$ plays an essential role.

**PERSONAL ASIDE:** Loop groups and Birkhoff factorizations appear in quantum cohomology theory at this point. $\hbar$ is the loop parameter. (M. Guest, “From Quantum Cohomology to Integrable Systems”, CUP 2008.)
More generally, if \( \dim H^2(M; \mathbb{C}) = r \), then we have a system of linear p.d.e. in variables \( q_1, \ldots, q_r \). A new source of difficulty appears: while an o.d.e. \( Py = 0 \) of order \( k \) always has a (local) solution space of dimension \( k \), there is no such dimension formula in terms of the orders \( k_1, k_2, \ldots \) of the operators \( P_1, P_2, \ldots \) of a system. In the situation of the quantum differential equation this dimension must be equal to \( \dim H^*(M; \mathbb{C}) \); this is a nontrivial property of quantum cohomology.

In fact, systems of linear p.d.e. with (nonzero, but) finite-dimensional local solution space are quite rare. They correspond to flat connections in vector bundles, or D-modules of finite rank.
Recall that a connection in a vector bundle on \((q_1, \ldots, q_r)\)-space may be expressed as \(\nabla = d + \omega\), where \(\omega\) is a matrix-valued 1-form. It is said to be flat if \(d\omega + \omega \wedge \omega = 0\). In the case of quantum cohomology (or a Frobenius manifold), the connection is called the Givental connection, or Dubrovin connection.

In general, if a connection form is written in terms of

\[ f_1(q_1, \ldots, q_r), f_2(q_1, \ldots, q_r), \ldots \]

then the flatness condition is a system of nonlinear p.d.e. for the functions \(f_1, f_2, \ldots\) and there is a close relation between this nonlinear p.d.e. and the original linear p.d.e. The theory of “integrable p.d.e.” amounts to studying this kind of relation. For example, the (nonlinear) KdV equation is related to the (linear) Schrödinger equation in exactly this way.
Quantum cohomology (in the sense of most of this talk, i.e. small quantum cohomology of Fano manifolds) fits into this framework, but only in a trivial way. This is because the entries of the Dubrovin connection form are just polynomials on $H^2(M;\mathbb{C})$. For small quantum cohomology, the WDVV equations are trivial. However, big quantum cohomology involves functions on $H^*(M;\mathbb{C})$, which are “known” only to the extent that they are solutions of the WDVV equations.
SUMMARY:

(moduli) space of holomorphic maps

\[ \rightsquigarrow \]

Gromov-Witten invariants (intersection theory)

\[ \rightsquigarrow \]

quantum differential equation (linear p.d.e.)

\[ \rightsquigarrow \]

WDVV equation (nonlinear p.d.e.)
SOME RECENT DEVELOPMENTS

In this section we mention some further developments related to quantum cohomology, mainly from the viewpoint of differential equations.
Stokes data of the quantum differential equation

The quantum differential equation for $M = \mathbb{CP}^n$ has a regular singularity at $q = 0$ and an irregular singularity at $q = \infty$.

Near $q = 0$ we have seen a nice series expansion, related to Gromov-Witten invariants.

Near $q = \infty$ we certainly have (local, possibly multivalued) solutions, but “series expansions at infinity” always diverge. This is the Stokes Phenomenon: such series represent only asymptotic expansions of solutions, and only on specific Stokes sectors of the complex plane.
Remarkably (and non-intuitively) the asymptotic expansion — which depends only on the equation — determines a unique solution on a given Stokes sector.

Two such solutions on overlapping sectors must differ by a constant matrix; this is called a Stokes matrix.

Analytically continuing a given solution one circuit around $q = \infty$, we pass through a finite number of sectors, so we see that the monodromy of that solution is (essentially) a product of Stokes matrices.
In general, Stokes matrices are difficult to compute, and their entries are usually highly transcendental, but in the case of the quantum differential equations we might expect them to contain geometric/physical information. This is indeed the case.

For $M = \mathbb{C}P^n$ the Stokes data reduces to $n$ real numbers $s_1, \ldots, s_n$. These were computed by Dubrovin for $n = 2$, and by Davide Guzzetti for general $n$, and the answer is surprisingly simple:

$$s_k = \binom{n + 1}{k}$$

Physicists had also arrived at this fact by “counting solitons” in the supersymmetric sigma model of $\mathbb{C}P^n$. Clearly something interesting is happening...
A further piece of information can be extracted from the differential equation, by comparing the Frobenius solution near $q = 0$ with (any of) the Stokes solutions near $q = \infty$. In the differential equations literature, the matrices which occur here are called connection matrices.

A remarkable geometrical interpretation of this was found by Hiroshi Iritani. It involves the “gamma class”

$$\hat{\Gamma}_M = \prod_{i=1}^{n} \Gamma(1 + x_i)$$

where $x_1, \ldots, x_n$ are the Chern roots of the complex tangent bundle $TM$, and where $\Gamma(1 + x_i)$ is interpreted as the (Taylor expansion of) the gamma function. This is very much in the spirit of Givental’s homological geometry.
Isomonodromic deformations

The Stokes data of the quantum differential equation can be approached most directly by converting the quantum differential equation (a p.d.e. in $q_1, \ldots, q_r$ with parameter $\hbar$) to an o.d.e. in $\hbar$ with parameters $q_1, \ldots, q_r$. This “trick” is made possible by homogeneity.

An important property of the o.d.e. in $\hbar$ is that it is isomonodromic: its monodromy data (Stokes and connection matrices) do not depend on $q_1, \ldots, q_r$. 
For small quantum cohomology this property is not very significant (as the WDVV equation is trivial here), but in the case of big quantum cohomology it means that the monodromy data gives \textit{conserved quantities} of solutions of the WDVV equations.

This provides another link with the theory of integrable systems. Indeed, Dubrovin proposed this as a general approach to the study of Frobenius manifolds.

There is a well-developed method — the Riemann Hilbert Method — for studying the relation between monodromy data of linear o.d.e. and solutions of nonlinear p.d.e.
Summary of differential equations so far:

LINEAR EQUATIONS:

small q.d.e $\subseteq$ big q.d.e

(equations for quantum cohomology)

NONLINEAR EQUATIONS:

(trivial) $\subseteq$ WDVV

(equations for Frobenius structure of quantum cohomology)

The left hand side is very easy; the right hand side is very difficult. We shall consider a situation of intermediate difficulty next.
The topological-antitopological fusion equations

The $tt^*$ equations are a nonlinear system which appears when we consider Frobenius manifolds “with Hermitian metric” (or “with real structure”). In this situation the nonlinear equations are enhanced:

NONLINEAR EQUATIONS:

\[ tt^* \text{ equations} \subseteq \text{“} tt^*-\text{WDVV equations} \text{”} \]

(equations for Frobenius structure with Hermitian metric of quantum cohomology)
The $tt^*$ equations were introduced by Sergio Cecotti and Cumrun Vafa in the context of supersymmetric quantum field theory. Dubrovin formulated these equations as an isomonodromic system, so the Riemann-Hilbert Method can be applied.

IDEA: A Frobenius manifold is a holomorphic object, but, if it has a real structure, then there is a “complex conjugate” Frobenius manifold, which is an antiholomorphic object. Now, a Frobenius manifold has, as part of its definition, a “holomorphic metric”. Thus, the real structure produces a Hermitian metric. The $tt^*$ equations are the equations for this metric — the $tt^*$ metric.
For Frobenius manifolds, e.g. big quantum cohomology, the tt* equations are extremely complicated. However, even after restricting to small quantum cohomology (where the WDVV equations are trivial), some nonlinear equations remain.

As Dubrovin pointed out, these nonlinear equations are very familiar to differential geometers: they are the equations for pluriharmonic maps from \((q_1, \ldots, q_r)\)-space to the Riemannian symmetric space \(\text{GL}_{n+1} \mathbb{R}/\text{O}_{n+1}\) (where \(n + 1\) is the dimension of the Frobenius manifold).
When \( r = 1 \), a pluriharmonic map is just a harmonic map. Here the theory of harmonic maps from surfaces into symmetric spaces (which was comprehensively developed in the 1980’s) provides an effective tool.

**PERSONAL ASIDE:** Harmonic maps from surfaces into symmetric spaces were studied by differential geometers in the 1970’s (E. Calabi, S.-S. Chern, J. Eells,...). In the 1980’s new methods were introduced: integrable systems (V. Zakharov-A. Shabat, K. Uhlenbeck,...); loop groups (G. Segal,...). Many geometers in the UK, Germany, Japan developed these methods further (U. Pinkall, F. Burstall, J. Dorfmeister,..., M. Guest-Y. Ohnita,...)
When $r = 1$, the tt* equations of Cecotti and Vafa are

$$\frac{\partial}{\partial t} \left( g \frac{\partial}{\partial t} g^{-1} \right) - [C, gC^\dagger g^{-1}] = 0$$

where

- $C = \text{(holomorphic) chiral matrix of the theory}$
- $C^\dagger = \text{conjugate-transpose of } C$,
- $g^{-1} = \text{Hermitian matrix representing the tt* metric}$

In the situation of quantum cohomology, $C$ is the matrix of quantum multiplication by a generator $b \in H^2(M; \mathbb{C})$. 
Let us now specialize to the case $M = \mathbb{C}P^n$.

Then the tt* metric can be written

$$g = \text{diag}(e^{-2w_0}, \ldots, e^{-2w_n})$$

where $w_0, \ldots, w_n$ are real valued functions of $q, \bar{q}$.

The homogeneity property of quantum cohomology implies that $w_i$ depends only on $|q|$. In view of this, it is convenient to make a change of variable of the form $q = \alpha t^\beta$, and then the tt* equations become

$$2(w_i)_{t\bar{t}} = -e^{2(w_{i+1}-w_i)} + e^{2(w_i-w_{i-1})}, \ i = 0, 1, \ldots, n$$

together with the extra condition $w_i + w_{n-i} = 0$.

This is a version of the periodic Toda equations. We call it the tt*-Toda equation.

(We interpret $w_{n+1}, w_{-1}$ here as $w_0, w_n$ respectively.)
Physically, a solution is a massive deformation of a conformal field theory, and the existence of such a deformation says something about that theory. Cecotti and Vafa made a series of conjectures about the solutions:

- there should exist (globally smooth) solutions \( w = w(|t|) \) on \( \mathbb{C}^* \)
- these solutions should be characterized by asymptotic data at \( t = 0 \) (the “ultra-violet point”; here the data is the chiral charges, essentially the holomorphic matrix \( C' \))
- these solutions should equally be characterized by asymptotic data at \( t = \infty \), (the “infra-red point”; here the data is the soliton multiplicities \( s_i \)).

Note: We will say more about solitons later.
Theorem:
(M. Guest - A. Its - C.-S. Lin, T. Mochizuki)
For each $N > 0$, there is a 1 : 1 correspondence between solutions of the tt*-Toda equations on $\mathbb{C}^*$ and 1-forms $\eta(z) \, dz$ on (the universal cover of) $\mathbb{C}^*$, where

$$
\eta(z) = \begin{pmatrix}
z^{k_0} \\
z^{k_1} \\
. . . \\
z^{k_n}
\end{pmatrix}
$$

- $k_i \in [-1, \infty)$
- $n + 1 + \sum_{i=0}^{n} k_i = N$
- $k_i = k_{n-i+1}$ for $i = 1, \ldots, n$

The variable $z$ is related to the variable $t$ of the tt*-Toda equations by $t = \frac{n+1}{N} \, z^{\frac{N}{n+1}}$. 
The $k_i$ have several interpretations:

- Asymptotics at $t = 0$: $w_i \sim -m_i \log |t|$ as $t \to 0$, where the $m_i$ are defined by $m_i - m_{i-1} + 1 = \frac{n+1}{N}(k_i + 1)$.

- $\frac{1}{\lambda} \eta(z) dz$ is a (normalized) DPW potential for the harmonic map corresponding to the solution ($\lambda = \hbar = $ loop parameter)

- $\eta(z) dz$ is a Higgs field for the harmonic bundle corresponding to the solution

These solutions can also be characterized by their asymptotics at $t = \infty$: 
**Theorem:** (M. Guest - A. Its - C.-S. Lin)

There is a 1 : 1 correspondence between solutions of the tt*-Toda equations on $\mathbb{C}^*$ with $n$-tuples of “Stokes parameters” $s = (s_1, \ldots, s_n)$ where $s_i$ is the $i$-th symmetric function of $e^{(2m_0+n)\frac{\pi \sqrt{-1}}{n+1}}$, $e^{(2m_1+n-2)\frac{\pi \sqrt{-1}}{n+1}}$, ..., $e^{(2m_n-n)\frac{\pi \sqrt{-1}}{n+1}}$.

As $t \to \infty$ we have, for each $k = 1, \ldots, n$,

$$\frac{4}{n+1} \left[ \frac{1}{2} (n-1) \right] \sum_{p=0}^{\left[ \frac{1}{2} (n+1) \right]} w_p \sin \frac{(2p+1)k\pi}{n+1} \sim s_k F(L_k|t|)$$

where

- $F(|t|) = \frac{1}{2} (\pi |t|)^{-\frac{1}{2}} e^{-2|t|}$
- $L_k = 2 \sin \frac{k}{n+1} \pi$
- $\left[ \frac{1}{2} (n+1) \right] = \frac{1}{2} (n+1)$ if $n$ is odd, $\frac{1}{2} n$ if $n$ is even
SOME APPLICATIONS

(of the above results, and the methods used to prove them — and some connections with physics)
Application 1: The Coxeter Plane

(in this section we use some Lie theory, mainly for $\mathfrak{g} = \mathfrak{sl}_{n+1}\mathbb{C}$, the Lie algebra of $(n + 1) \times (n + 1)$ complex matrices of trace 0)

Let $\mathfrak{g}$ be a complex simple Lie algebra, with corresponding simply-connected Lie group $G$. Let $\alpha_1, \ldots, \alpha_l \in \mathfrak{h}^*$ be a choice of simple roots of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$. (Thus $\text{rank } \mathfrak{g} = \dim \mathfrak{h} = l.$)
The Weyl group $W$ is the finite group generated by the reflections $r_{\alpha}$ in all root planes $\ker \alpha$, $\alpha \in \Delta$.

The Coxeter element is the element $\gamma = r_{\alpha_1} \cdots r_{\alpha_l}$ of $W$. Its order is called the Coxeter number of $\mathfrak{g}$, and we denote it by $s$.

Fact (Kostant): The Coxeter element $\gamma$ acts on the set of roots $\Delta$ with $l$ orbits, each containing $s$ elements.

(if $\mathfrak{g} = \mathfrak{sl}_{n+1}\mathbb{C}$ then $l = n$, $s = n + 1$, and $W$ is the permutation group on $n + 1$ objects)
The Coxeter Plane is the result of projecting Δ orthogonally onto a certain real plane in $\mathfrak{h}^*$. (See M. Guest and N.-K. Ho, Kostant, Steinberg, and the Stokes matrices of the tt*-Toda equations, Sel. Math. New Ser. 2019 for several definitions.)

E.g. the Coxeter Plane for $\mathfrak{g} = \mathfrak{sl}_5 \mathbb{C}$:

(there are 20 roots $x_i - x_j$, $0 \leq i \neq j \leq 4$, and the Coxeter element acts by the permutation (43210); there are $l = 4$ orbits, each containing $s = 5$ elements)
**Theorem:** (M. Guest - N.-K. Ho)

(i) The Coxeter Plane is a diagram of the Stokes sectors for\(^\dagger\) the tt*-Toda equation.

(ii) The Stokes matrices can be computed Lie-theoretically in terms of a Lie group element

\[ M^{(0)} = C(s_1, \ldots, s_l) \in \text{SL}_{n+1} \mathbb{C} \]

where \( C \) is a “Steinberg cross-section”.

Fact (Steinberg): If \( A \in \text{SL}_{n+1} \mathbb{C} \) satisfies “minimal poly. of \( A = \) characteristic poly. of \( A \)” then \( \exists! (s_1, \ldots, s_l) \) s.t. \( C(s_1, \ldots, s_l) \) is conjugate to \( A \).

\(^\dagger\)(more precisely, for the isomonodromic connection associated by homogeneity to the tt*-Toda equation; equivalently, for the isomonodromic connection associated by homogeneity to the quantum differential equation)
Moreover, the space of solutions also has a Lie-theoretic interpretation:

Recall that the solutions are parametrized by \((n + 1)\)-tuples \((m_0, \ldots, m_n)\) satisfying certain inequalities. These inequalities define a convex polytope.

Let us write

\[
m = \text{diag}(m_0, m_1, \ldots, m_n)
\]

\[
\rho = \text{diag}(\frac{n}{2}, \frac{n}{2} - 1, \ldots, -\frac{n}{2})
\]

Then the convex polytope given by the points

\[
\frac{2\pi \sqrt{-1}}{n + 1}(m + \rho)
\]

is the Fundamental Weyl Alcove of the Lie algebra.
The quantum cohomology of $\mathbb{C}P^n$ corresponds to one of these solutions.

Q: which one?

A: the solution given by $m = -\rho$

(i.e. the origin of the Fundamental Weyl Alcove).

This suggests the next question:

Q: are there any more “interesting” solutions? 

A: yes!

(in fact Cecotti and Vafa proposed that the physically meaningful solutions are given by those $m$ for which all $s_k \in \mathbb{Z}$; they gave some examples, but these solutions have not been investigated in detail)

(Recall that $s_k = \binom{n+1}{k}$ for the solution which corresponds to the quantum cohomology of $\mathbb{C}P^n$.)
Application 2: Particles and polytopes

In this section we show how the Coxeter Plane and the tt*-Toda equations give a mathematical foundation for certain field theory models proposed by physicists in the 1990’s.
The Coxeter Plane has appeared (implicitly) in articles on Toda field theory:


In this “toy model” the authors proposed (amongst other things) the correspondence

\[ \text{particle} \leftrightarrow \text{orbit of root in Coxeter Plane} \]
\[ \text{mass of particle} \leftrightarrow \text{distance of root from origin} \]

(if \( \mathfrak{g} = \mathfrak{sl}_{n+1} \mathbb{C} \) the mass of the particle corresponding to the orbit of the root \( x_i - x_j \) is
\[ 2 \sin |i - j| \frac{\pi}{n+1} \])

They checked that these proposals (as well as the other things) were consistent with the expected properties of a field theory.
A variant of this proposal was made in


In these “polytopic models”, a finite-dimensional representation $\theta$ of the Lie algebra $\mathfrak{g}$ on a vector space $V$ is chosen, and the “polytope” is the polytope in $\mathfrak{h}^*$ spanned by the weights of the representation. The weight vectors (in $V$) are taken to be the vacua of the theory. In this theory, “solitonic particles” tunnel between vacua: a soliton connects two vacua $v_i, v_j$ if and only if the corresponding weights $\lambda_i, \lambda_j$ differ by a single root, i.e. $\lambda_i - \lambda_j \in \Delta$. The physical characteristics of this particle are those of the root.
This discussion is purely algebraic (there is no differential equation). However, the polytopic models include certain Landau-Ginzburg models. The quantum cohomology of $\mathbb{C}P^n$ is of this type, with: $\theta = \lambda_{n+1}$ (standard representation of $\mathfrak{sl}_{n+1}\mathbb{C}$). Thus we can expect a role for solitons in the quantum cohomology of $\mathbb{C}P^n$.

The solitons are illustrated below for $\mathfrak{sl}_4\mathbb{C}$.

The first part shows the projections of the weights $x_0, x_1, x_2, x_3$. The second part shows (as heavy lines) the four solitons of type [01] (with mass $2 \sin \frac{\pi}{4} = \sqrt{2}$). The third part shows the two solitons of type [02] (with mass $2 \sin \frac{\pi}{2} = 2$). In this example, any two vacua are connected by a soliton.
Each solution of the $tt^*$-Toda equation (e.g. that with $m = -\rho$) is associated to a field theory. That theory fits into this framework as follows.

**Corollary:** (of the proof of the theorem on asymptotics at $t = \infty$)

The linear combination on the left hand side of

$$-\frac{4}{n+1} \sum_{p=0}^{\left[\frac{1}{2}(n-1)\right]} w_p \sin \left(\frac{(2p+1)k\pi}{n+1}\right) \sim s_k F(L_k|t|)$$

corresponds to a certain$^\dagger$ basis vector of $\mathfrak{h}$ (or $\mathfrak{h}^*$) associated to an orbit of the Coxeter group.

Thus we can say that the Stokes parameter $s_k$ is naturally associated to the $k$-th orbit, or particle. Physicists call $s_k$ the soliton multiplicity.

In the above example we chose $\theta = \lambda_{n+1}$. If we choose

$$\theta = \wedge^k \lambda_{n+1}$$

we obtain a different polytopic model. It turns out that the quantum cohomology of the Grassmannian $Gr_k(C^{n+1})$ is of this type.

The solitons are illustrated below for $\mathfrak{sl}_4\mathbb{C}$. The first part shows the projections of the weights $x_i + x_j$ with $0 \leq i \neq j \leq 3$. The second part shows the four solitons of type $[01]$ (with mass $\sqrt{2}$). The third part shows the four solitons of type $[02]$ (with mass 2).
A more complicated example: $\theta = \wedge^3 \lambda_6$

The quantum cohomology of the Grassmannian $Gr_3(\mathbb{C}^6)$ is of this type.

<table>
<thead>
<tr>
<th>particle</th>
<th>$[x_0 - x_1]$</th>
<th>$[x_0 - x_2]$</th>
<th>$[x_0 - x_3]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass $L_k$</td>
<td>$2 \sin \frac{\pi}{6} = 1$</td>
<td>$2 \sin \frac{2\pi}{6} = \sqrt{3}$</td>
<td>$2 \sin \frac{3\pi}{6} = 2$</td>
</tr>
<tr>
<td>mult. $s_k$</td>
<td>$s_1 = s_5 = \binom{6}{1}$</td>
<td>$s_2 = s_4 = \binom{6}{2}$</td>
<td>$s_3 = \binom{6}{3}$</td>
</tr>
</tbody>
</table>
Cecotti and Vafa used this to give a physical argument for an “equivalence”

\[ \wedge^k QH^*(\mathbb{C}P^n) \approx QH^*(Gr_k(\mathbb{C}^{n+1})) \]

(more precisely, an equivalence of underlying field theories). This tt* argument was explained in


Later, mathematicians gave proofs of versions of this isomorphism (which they regard as a special case of the quantum Satake isomorphism, or abelian-nonabelian correspondence). E.g.

V. Golyshev and L. Manivel, Quantum cohomology and the Satake isomorphism, arXiv:1106.3120
Our Lie-theoretic description of the solutions of the \( \text{tt}^* \)-Toda equations supports the original physics argument, because

solution with \( m = -\rho \)

\[
\begin{align*}
\theta &= \lambda_{n+1} \\
\xrightarrow{\text{QH}} QH^*(\mathbb{CP}^n)
\end{align*}
\]

solution with \( m = -\rho \)

\[
\begin{align*}
\theta &= \wedge^k \lambda_{n+1} \\
\xrightarrow{\text{QH}} QH^*(Gr_k(\mathbb{C}^{n+1}))
\end{align*}
\]

i.e. the same solution of the \( \text{tt}^* \)-Toda equations gives both \( QH^*(\mathbb{CP}^n) \) and \( QH^*(Gr_k(\mathbb{C}^{n+1})) \).

The Stokes matrices of the respective quantum differential equations are different (they can be read off from \( M^{(0)} \) and \( \wedge^k M^{(0)} \) respectively). But the Stokes parameters \( s_k = \binom{n+1}{k} \) are the same for \( QH^*(\mathbb{CP}^n) \) and \( QH^*(Gr_k(\mathbb{C}^{n+1})) \).
Application 3: Minimal models

Recall that the “Higgs fields”

\[
\eta(z) = \begin{pmatrix}
\vdots \\
z^{k_0} \\
z^{k_1} \\
\vdots \\
z^{k_n}
\end{pmatrix}
\]

(with \(k_i \in [-1, \infty)\), \(n + 1 + \sum_{i=0}^{n} k_i = N\), \(k_i = k_{n-i+1}\) for \(i = 1, \ldots, n\)) parametrize solutions of the tt*-Toda equations.

In this section we consider \(\eta dz\) with \(k_i \in \mathbb{Z}_{\geq 0}\) and assume that \(N\) is coprime to \(k = \sum_{i=0}^{n} k_i\).

Thus we move away from the tt*-Toda equations (but we note that the Higgs fields with \(k_i = k_{n-i+1}\) for \(i = 1, \ldots, n\) form a dense subset of solutions of the tt*-Toda equations).
The authors of

L. Fredrickson and A. Neitzke, From $S^1$-fixed points to $W$-algebra representations, 
arXiv:1709.06142

study a certain moduli space $M_{K,N}$ of Higgs fields with a $\mathbb{C}^*$-action whose fixed points are the $\eta dz$. Quoting from this article:

“We ... exhibit a curious 1-1 correspondence between these fixed points and certain representations of the vertex algebra $W_K$; in particular we have $12\mu = K - 1 - c_{eff}$, where $12\mu$ is a ... norm of the Higgs field, and $c_{eff}$ is the effective Virasoro central charge.”

“The formula $12\mu = K - 1 - c_{eff}$ is puzzling. Why should $W_K$ and $M_{K,N}$ have anything to do with one another?”

†The published version (Math. Proc. Camb. Phil. Soc., online) has a different title: Moduli of wild Higgs bundles on $CP^1$ with $\mathbb{C}^\times$-actions. (The abstract of the published version states the formula erroneously as $\mu = K - 1 - c_{eff}/12$.)
The authors remark that $M_{K,N}$ is the moduli space of a certain four-dimensional supersymmetric quantum field theory, namely Argyres-Douglas theory of type $(A_{K-1}, A_{N-1})$, and that $W_K$ is known to be associated to this theory. They propose:

“One may hope that the explanation ... will be found in that theory. So far we have not found such an explanation; the correspondence seems to us to be the tip of an iceberg of unknown size.”

As an application of our Lie-theoretic Stokes formula

$$M^{(0)} = C'(s_1, \ldots, s_l) \in \text{SL}_{n+1}\mathbb{C}$$

we shall give a mathematical explanation — a direct path from $\eta dz$ to the representation.

Recall that the irreducible positive energy representations of the affine Kac-Moody algebra $\hat{\mathfrak{sl}}_{n+1}\mathbb{C}$ of level $k$ ($\in \mathbb{N}$) are parametrized by dominant weights $(\Lambda, k)$, where $\Lambda$ is a dominant weight of $\mathfrak{sl}_{n+1}\mathbb{C}$ of level $k$.

Let $P_+$ be the set of dominant weights of $\mathfrak{sl}_{n+1}\mathbb{C}$, and $P_k = \{ \text{dominant weights of level } k \}$.

It is well known that $P_k + \rho = P_+ \cap (k + n + 1)\hat{A}$ where $\hat{A}$ denotes the interior of the Weyl alcove $A$.

Let $\hat{A}_k = \left( \frac{1}{k+n+1} P_+ \right) \cap \hat{A}$. Let $\theta : \hat{A}_k \to P_k + \rho$ be the identification given by

$\theta(v) = (k + n + 1)v \in P_+ \cap (k + n + 1)\hat{A} = P_k + \rho$. 

Recall that the Stokes data (of the Higgs field $\eta dz$) is represented by the matrix $M^{(0)}$. It follows from the assumption $k_i \in \mathbb{Z}_{\geq 0}$ that $M^{(0)}$ is semisimple; in fact it is conjugate to the diagonal matrix

$$M^{(0)}_{\text{diag}} = e^{\frac{2\pi \sqrt{-1}}{n+1} (m+\rho)}$$

where $\frac{2\pi \sqrt{-1}}{n+1} (m + \rho)$ is in the Fundamental Weyl Alcove of $\mathfrak{sl}_{n+1} \mathbb{C}$.

**Lemma:** Let $2\pi \sqrt{-1} \epsilon_1, \ldots, 2\pi \sqrt{-1} \epsilon_n$ denote the basic weights of $\mathfrak{sl}_{n+1} \mathbb{C}$. Then:

$$\frac{N}{n+1} (m + \rho) = \rho + \sum_{i=1}^{n} k_i \epsilon_i.$$

**Proof:** This is equivalent to the relation

$$m_{i-1} - m_i + 1 = \frac{n+1}{N} (k_i + 1)$$

which defines the $m_i$ in terms of the $k_i$. \qed

It follows that $\theta(\frac{1}{n+1} (m + \rho)) = \rho + \sum_{i=1}^{n} k_i \epsilon_i$.

Thus, from the Stokes data $M^{(0)}$ we obtain the positive energy representation with dominant weight $(\sum_{i=1}^{n} 2\pi \sqrt{-1} k_i \epsilon_i, k)$. 

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It is well known (Bouwknegt and Schoutens) that the $W$-algebra $W_{n+1}$ intertwines with any such representation, and that the effective central charge is given by the formula

$$c_{eff} = n - 12 \frac{n+1}{N} \left| \sum_{i=1}^{n} k_i \epsilon_i - \frac{k}{n+1} \rho \right|^2.$$

By the lemma we have $\sum_{i=1}^{n} k_i \epsilon_i - \frac{k}{n+1} \rho = \frac{N}{n+1} m$ so

$$c_{eff} = n - 12 \frac{N}{n+1} |m|^2.$$

This is the formula of Fredrickson and Neitzke which Higgs fields and representations of $W_{n+1}$. Our construction shows that the Stokes data of the Higgs field is responsible for the relation.

Remark: For fixed $n+1$ and $N$ there are a finite number of representations. These constitute the “$(n + 1, N) \ W_{n+1}$ minimal model”.
Thank you!

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http://www.f.waseda.jp/martin/