DETECTING EXOTIC STRUCTURES
VIA THE PONTRJAGIN-THOM CONSTRUCTION

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Abstract. Kreck and Stolz recently exhibited exotic structures on a family of seven dimensional homogeneous spaces which are quotients of the compact Lie group $SU_3$. We observe that there is an invariant obtained via the Pontrjagin-Thom construction which detects these exotic structures in many cases.

In their paper [5] Kreck and Stolz exhibited new examples of homeomorphic but non-diffeomorphic homogeneous spaces of dimension 7. These spaces, the so called Wallach spaces, are all stably parallelizable, and in fact for each one there is a canonical trivialization of the (stable) tangent bundle. Hence, via the Pontrjagin-Thom construction (see [7]), each such space $N$ gives rise to an element $[N]$ of the stable 7-stem $\pi_7^s \cong \mathbb{Z}/240$. Although this construction depends a priori on the tangent bundle and therefore on the differentiable structure of $N$, the invariant $[N]$ has in the past proved to be disappointingly rigid (cf. [3] and Commentary of [2]). It is the purpose of this note to point out that in fact the invariant $[N]$ reflects surprisingly well the exotic behaviour discovered by Kreck and Stolz.

First, we show that the invariant $5[N]$ is a diffeomorphism invariant for Wallach spaces (corollary 3). Second, we show that $5[N]$ detects the exotic differentiable structure in many of the examples of Kreck and Stolz. Our results are purely experimental observations; they are direct consequences of the classification theorems of [5], together with the computation of $[N]$. While this proof “after the fact” is itself of no consequence, we believe that the phenomenon exhibited may be worthy of further study.

We begin by recalling the definition of the Wallach spaces $N_{k,l}$, where $k, l$ are non-zero coprime integers. Let $S_{k,l}$ be the subgroup of the group $SU_3$ consisting of diagonal matrices with diagonal entries $z^k, z^l, z^{-(k+l)}$, where $z$ is any unit complex number. Then $N_{k,l}$ is defined to be the coset space $SU_3/S_{k,l}$. The manifold $N_{k,l}$ is a simply-connected 7-dimensional homogeneous space whose integral cohomology is as follows: $H^i(N_{k,l}) \cong \mathbb{Z}$ for $i = 0, 2, 5, 7$; $H^i(N_{k,l}) = 0$ for $i = 1, 3, 6$; $H^4(N_{k,l}) \cong \mathbb{Z}/N(k,l)$, where $N(k,l) = k^2 + kl + l^2$. For this and for further information (especially differential geometric), we refer to [4],[12],[13].

The result of Kreck and Stolz is:

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Theorem 1 (Kreck and Stolz, [5]). Assume that \( k, l, \tilde{k}, \tilde{l} \) are non-zero integers with \((k, l) = 1, (\tilde{k}, \tilde{l}) = 1\). Then

1. \( N_{k,l} \) is homeomorphic to \( N_{k,\tilde{l}} \) if and only if \( N(k, l) = N(\tilde{k}, \tilde{l}) \) and \( kl(k + l) \equiv \tilde{k}\tilde{l}(\tilde{k} + \tilde{l}) \mod 2^5 \cdot 3 \cdot N(k, l) \).

2. \( N_{k,l} \) is diffeomorphic to \( N_{k,\tilde{l}} \) if and only if \( N(k, l) = N(\tilde{k}, \tilde{l}) \) and \( kl(k + l) \equiv \tilde{k}\tilde{l}(\tilde{k} + \tilde{l}) \mod 2^5 \cdot 3 \cdot 7^2 \cdot N(k, l) \), where \( \lambda(k, l) = 0 \) if \( N(k, l) \) is divisible by 7 and \( \lambda(k, l) = 1 \) otherwise. \( \square \)

It is elementary that \( N_{k,l} \) and \( N_{k,\tilde{l}} \) are diffeomorphic if \( S_{k,l} \) and \( S_{k,\tilde{l}} \) are conjugate in \( SU_3 \), i.e. if \( \{k, l, -(k + l)\} = \{\tilde{k}, \tilde{l}, -(\tilde{k} + \tilde{l})\} \). However, Kreck and Stolz list fourteen further examples where \( N_{k,l} \) and \( N_{k,\tilde{l}} \) are homeomorphic. In three of these the spaces are actually diffeomorphic, but the remaining eleven cases provide examples of Wallach spaces which are homeomorphic but non-diffeomorphic. The simplest example of this exotic behaviour is given by \( k = -56788, l = 5227, \tilde{k} = -42652, \tilde{l} = 61213 \). (Extensive computer calculations were needed to produce these fourteen examples. According to [5], sixteen additional examples of homeomorphic pairs have been found, three of which are diffeomorphic and thirteen of which are non-diffeomorphic. Whether there are infinitely many such pairs is still an open question.)

Now we turn to the Pontrjagin-Thom construction. If \( G \) is an oriented compact Lie group, and \( H \) is any abelian subgroup, then it is easy to show that the left invariant framing of \( G \) induces a stable framing of the homogeneous space \( G/H \) (see [6]). The Pontrjagin-Thom construction associates to this framing an element \([G/H] \) in the group \( \pi^s_d \), where \( d \) is the dimension of \( G/H \). For the Wallach spaces \( N_{k,l} \) we obtain elements \([N_{k,l}] \) in \( \pi^s_d \cong \mathbb{Z}/240 \). Now, there is a homomorphism \( e_C : \pi^s_d \rightarrow \mathbb{Q}/\mathbb{Z}, \) the complex \( e \)-invariant. It is known (see [1]) that this homomorphism is injective, and that a generator \( \iota \) of \( \pi^s_d \) may be chosen so that \( e_C(\iota) = 1/240 \). This enables us to compute \([N_{k,l}] \) explicitly:

**Proposition 2.** An orientation of \( SU_3 \) may be chosen so that \([N_{k,l}] = kl(k + l)\iota \in \pi^s_d \).

**Proof.** It suffices to compute \( e[N_{k,l}] \). To do this, we use the method of [6]. Consider the principal circle bundle \( p_{k,l} : N_{k,l} = SU_3/S_{k,l} \rightarrow SU_3/T \) where \( T \) is the subgroup of \( SU_3 \) consisting of diagonal matrices. Since \( T \) is abelian, our stable framing of \( N_{k,l} \) actually induces a stable framing of \( SU_3/T \), hence it is of the type considered in [6]. According to proposition 2.1 of [6] we have

\[
e_C[N_{k,l}] = -(1/240)\langle c_1(L_{k,l})^3, F \rangle
\]

where \( c_1(L_{k,l}) \) is the Chern class of the complex line bundle \( L_{k,l} \) associated to \( p_{k,l} \), and where \( F \) is the fundamental homology class of \( SU_3/T \) (with respect to the orientation induced by the stable framing). The integral cohomology ring of \( SU_3/T \) is well known to be generated by classes \( w, x, y \in H^2(SU_3/T) \) subject to the condition that all symmetric polynomials in these classes are zero. (If \( SU_3/T \) is viewed as the “flag manifold” consisting of all triples of orthogonal lines in \( \mathbb{C}^3 \), the classes \( w, x, y \) are the Chern classes of the three tautologous complex line bundles on this space.) It follows that \( x^3 = y^3 = 0 \) and that \( H^3(SU_3/T) \) is generated by \( x^2y = -xy^2 \). One has (lemma 4.2 of [5]) \( c_1(L_{k,l}) = -lx + ky \). Hence \( c_1(L_{k,l})^3 = (lx + ky)^3 = -3kl(k + l)xy^2 \). With a suitable choice of orientation, \( \langle xy^2, F \rangle = 1 \). Thus, \( e_C[N_{k,l}] = kl(k + l)/240 \). This completes the proof. \( \square \)
The extent to which $[N_{k,l}]$ reflects the exotic behaviour of the spaces $N_{k,l}$ may now be read off from theorem 1. Let $N_{k,l}, N_{\tilde{k},\tilde{l}}$ be homeomorphic, and let $T = kl(k+l) - \tilde{k}\tilde{l}(\tilde{k}+\tilde{l})$ as in [5]. Thus, we are assuming that $N(k,l) = N(\tilde{k},\tilde{l})$ and that $T$ is divisible by $2^3 \cdot 3 \cdot N(k,l)$. Let $Q = T/2^3 \cdot 3 \cdot N(k,l)$. Then by theorem 1 we have:

(I) $N_{k,l}, N_{\tilde{k},\tilde{l}}$ are diffeomorphic if and only if $T \equiv 0 \mod 2^5 \cdot 3 \cdot 7^{\lambda(k,l)} \cdot N(k,l)$, i.e. if and only if $Q \equiv 0 \mod 2^2 \cdot 7^{\lambda(k,l)}$.

By proposition 2 we have

(II) $[N_{k,l}] = [N_{\tilde{k},\tilde{l}}]$ if and only if $T \equiv 0 \mod 2^4 \cdot 3 \cdot 5$.

In table 1, we list for each of the fourteen homeomorphic pairs in the table of [5] the values of $Q \mod 2^2 \cdot 7^{\lambda(k,l)}$, $T \mod 240$, and $5T \mod 240$, respectively.

<table>
<thead>
<tr>
<th></th>
<th>$Q \mod 2^2 \cdot 7^{\lambda(k,l)}$</th>
<th>$T \mod 240$</th>
<th>$5T \mod 240$</th>
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<tr>
<td>1</td>
<td>3</td>
<td>72</td>
<td>120</td>
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<tr>
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<tr>
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</table>

From columns 2 and 3 we see that for seven of the eleven pairs of homeomorphic, non-diffeomorphic Wallach spaces, the invariant $[N_{k,l}]$ distinguishes the exotic differentiable structures. Column 4 shows that in fact the invariant $5[N_{k,l}]$ suffices to distinguish them.
Conversely, we have:

**Corollary 3.** Let \( N_{k,l}, N_{\tilde{k},\tilde{l}} \) be diffeomorphic Wallach spaces. Then \( 5[N_{k,l}] = 5[N_{\tilde{k},\tilde{l}}] \).

**Proof.** This follows from conditions (I) and (II). Namely, if \( T \equiv 0 \mod 2^5 \cdot 3 \cdot 7^\lambda(k,l) \cdot N(k,l) \), then \( 5T \equiv 0 \mod 2^5 \cdot 3 \cdot 5 \). □

In fact, for the three pairs of diffeomorphic Wallach spaces in table 1 (numbers 7, 10 and 14), it is true that \([N_{k,l}] = [N_{\tilde{k},\tilde{l}}]\). One might therefore ask whether corollary 3 could be strengthened to say that \([N_{k,l}]\) is a diffeomorphism invariant. That this is not true is demonstrated by the following example, communicated to us by Stolz. Take \( k = -2646309363, \ l = -667411748, \ \tilde{k} = -2524823811, \ \tilde{l} = 3368004028 \) (this is one of the sixteen homeomorphic pairs not listed in the table 1). Then \( N_{k,l} \) is diffeomorphic to \( N_{\tilde{k},\tilde{l}} \), but \( [N_{k,l}] = 156\iota \) and \( [N_{\tilde{k},\tilde{l}}] = 204\iota \).

It is natural to look for other stably parallelizable manifolds which might exhibit behaviour similar to that of the Wallach spaces. Compact semi-simple Lie groups provide rather trivial examples. These have canonical framings, namely their left invariant framings. Moreover, the fact that homotopy equivalent groups of this type are known to be isomorphic (see [10],[11]) shows both that the Pontrjagin-Thom invariant is a diffeomorphism invariant and that there is no exotic behaviour. The exotic 7-spheres discovered by Milnor (see [8]), which are \( S^3 \)-bundles over \( S^4 \), may be expected to provide further examples. However, it is not clear that these possess canonical framings, and in any case the methods of [6] do not apply to compute the \( \iota \)-invariant, as the bundles are not sphere bundles of quaternionic line bundles (except in the case of the standard \( S^7 \)). In the case of a hypersurface in a sphere (see [9]), there is a natural framing given by the normal direction, but such examples are probably not sufficiently rigid to give rise to the behaviour observed for the Wallach spaces.

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References


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