ON STACKED TRIANGULATED MANIFOLDS

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Abstract. We prove two results on stacked triangulated manifolds in this paper: (a) every stacked triangulation of a connected manifold with or without boundary is obtained from a simplex or the boundary of a simplex by certain combinatorial operations; (b) for a connected closed manifold M of dimension $d \geq 4$, if the ith homology group vanishes for $1 < i < d-1$, then any tight triangulation of M is stacked. These results give affirmative answers to questions posed by Novik and Swartz and by Effenberger.

1. Introduction

Stacked triangulations of spheres are of fundamental, in particular in the study of convex polytopes and triangulations of spheres. Recently, the notion of stackedness was extended to triangulations of manifolds in [MN]. In this paper, we prove two results on stacked triangulations of manifolds.

We say that a simplicial complex Δ is a *triangulation* of a manifold M if its geometric carrier $|\Delta|$ is homeomorphic to M. A triangulation of a d-manifold with non-empty boundary is said to be *stacked* if all its interior faces have dimension $\geq d-1$. A triangulation of a closed d-manifold is said to be *stacked* if it is the boundary of a stacked triangulation of a $(d+1)$-manifold. A triangulation of a d-manifold is said to be *locally stacked* if each vertex link is a stacked triangulation of the $(d-1)$-sphere or the $(d-1)$-ball.

Kalai [Ka] proved that, for $d \geq 4$, every locally stacked triangulation of a connected closed d-manifold can be obtained from the boundary of a $(d+1)$-simplex by certain combinatorial operations. This result does not hold for 3-manifolds since there are triangulations of 3-manifolds which are locally stacked but cannot be obtained by their construction (see e.g. [BD2]). On the other hand, since the stackedness and the locally stackedness are equivalent in dimension ≥ 4 [BD3, MN], Kalai's result also characterizes stacked triangulations of connected closed manifolds of dimension ≥ 4. We give a similar characterization for stacked triangulations of manifolds with boundary of dimension ≥ 2 (Theorem 3.11). As a consequence, we generalize the result of Kalai to stacked triangulations of closed manifolds of dimension ≥ 2 (Corollary 3.12). This result and a recent result of Bagchi [Ba] solve a question posed by Novik and Swartz [NS, Problem 5.3].

Our second result is about an equivalence of tightness and tight-neighborliness. Let $\widetilde{H}_i(\Delta;\mathbb{F})$ be the ith reduced homology group of a topological space (or a simplicial complex) Δ with coefficients in a field $\mathbb{F}$. The number $\beta_i(\Delta;\mathbb{F}) := \dim_{\mathbb{F}} \widetilde{H}_i(\Delta;\mathbb{F})$ is called the *ith Betti number* of Δ with respect to $\mathbb{F}$. For a simplicial complex Δ on the vertex set V, we write $\Delta[W] = \{\alpha \in \Delta : \alpha \subset W\}$ for its induced subcomplex on $W \subseteq V$. A simplicial complex Δ on the vertex set V is said to be $\mathbb{F}$-*tight* if the natural map $\widetilde{H}_i(\Delta[W];\mathbb{F}) \to \widetilde{H}_i(\Delta;\mathbb{F})$ induced by the inclusion map is injective for all $W \subseteq V$ and for all $i \geq 0$. See [Kü, KL] for background and motivations of tightness. A simplicial complex Δ is said to be *neighborly* if each pair of vertices form a face. Note that $\mathbb{F}$-tight connected simplicial complex must be neighborly.

An n-vertex triangulation Δ of a connected closed manifold of dimension $d \geq 3$ is said to be *tight-neighborly* if $\binom{n-d-1}{2} = \binom{d+2}{2}\beta_1(\Delta; \mathbb{Z}/2\mathbb{Z})$. This condition is known to be equivalent to saying that Δ is stacked and neighborly (cf. Propositions 3.13, 3.14). Tight-neighborliness was introduced by Lutz, Sulanke and Swartz. They conjectured that tight-neighborly triangulations are $(\mathbb{Z}/2\mathbb{Z})$-tight [LSS, Conjecture 13]. The conjecture was solved by Effenberger [Ef, Corollary 4.4] in dimension ≥ 4 and by Burton, Datta, Singh and Spreer [BDSS, Corollary 1.3] in dimension 3. On the other hand, Effenberger [Ef, Question 4.5] asked if the converse of this property holds for triangulations of connected sums of S^{d-1}-bundles over S^1 when $d \geq 4$.

We answer Effenberger's question affirmatively (Corollary 4.4). This result and Effenberger's result say that, for triangulations of connected sums of S^{d-1}-bundles over S^1 with $d \geq 4$, tightness is equivalent to tight-neighborliness. Also, since tight-neighborly triangulations are vertex minimal triangulations, the result solves a special case of a conjecture of Kühnel and Lutz [KL, Conjecture 1.3] which states that every tight combinatorial triangulation is vertex minimal.

This paper is organized as follows. In Section 2, we give few basic definitions. In Section 3, we present a combinatorial characterization of stacked triangulations of manifolds with and without boundary. In Section 4, we study the stackedness of tight triangulations.

2. Preliminaries

Recall that a *simplicial complex* is a collection of finite sets (sets of *vertices*) such that every subset of an element is also an element. For $i \geq 0$, the elements of size $i+1$ are called the *i-faces* (or *i-simplices* or *faces of dimension i*) of the complex. The empty set $\emptyset$ is a face (of dimension -1) of every simplicial complex. Let $f_i(\Delta)$ be the number of its i-faces of Δ. For a simplicial complex Δ, the maximum of k such that Δ has a k-simplex is called the *dimension* of Δ and is denoted by $\dim(\Delta)$. A maximal face (under inclusion) in Δ is also called a *facet* of Δ. If σ is a face of Δ then the *link* of σ in Δ is the subcomplex

$$\mathrm{lk}_\Delta(\sigma) = \{\tau \setminus \sigma : \sigma \subseteq \tau \in \Delta\}.$$

For $d \geq 0$, if U is a set of $d+1$ elements then the simplicial complex $\overline{U}$ consists of all the subsets of U triangulates the d-ball and said to be the *standard d-ball.*

All simplicial complexes here are finite. For a field $\mathbb{F}$, a simplicial complex S of dimension d is said to be an *$\mathbb{F}$-homology d-sphere* if, for each face σ of dimension $i \geq -1$, $\mathrm{lk}_S(\sigma)$ has same $\mathbb{F}$-homologies as the $(d-i-1)$-sphere. A simplicial complex B of dimension d is said to be an *$\mathbb{F}$-homology d-ball* if (i) B has trivial reduced $\mathbb{F}$-homologies, (ii) for each face σ of dimension $i \leq d-1$, the reduced $\mathbb{F}$-homologies of $\mathrm{lk}_B(\sigma)$ are trivial or same as those of the $(d-i-1)$-sphere and (iii) the boundary $\partial B = \{\sigma \in B : -1 < \dim(\sigma) < d \text{ and } \widetilde{H}_{d-\dim(\sigma)-1}(\mathrm{lk}_B(\sigma); \mathbb{F}) = 0\} \cup \{\emptyset\}$ is an $\mathbb{F}$-homology $(d-1)$-sphere. A simplicial complex is said to be an *$\mathbb{F}$-homology d-manifold* if each vertex link is either an $\mathbb{F}$-homology $(d-1)$-sphere or a $(d-1)$-ball. Note that a triangulation of a d-manifold is a homology d-manifold.

By a *homology manifold/ball/sphere* we shall mean an $\mathbb{F}$-homology manifold/ball/sphere for some field $\mathbb{F}$. As before, the *boundary* of a homology d-manifold Δ is

$$\partial\Delta = \{\sigma \in \Delta : -1 < \dim(\sigma) < d \text{ and } \widetilde{H}_{d-\dim(\sigma)-1}(\mathrm{lk}_\Delta(\sigma); \mathbb{F}) = 0\} \cup \{\emptyset\}.$$

If $\partial\Delta = \{\emptyset\}$, then Δ is called a *closed homology d-manifold* (or a *homology d-manifold without boundary*), otherwise Δ is called a *homology d-manifold with boundary.* If Δ is a homology d-manifold with boundary, then $\partial\Delta$ becomes a closed homology $(d-1)$-manifold. We note the following easy fact.

Lemma 2.1. *Let Δ be a homology d-manifold with boundary. If Δ has a trivial reduced homologies then Δ is a homology d-ball.*

Proof. It is clear that Δ satisfies conditions (i) and (ii) of homology balls. The fact that $\partial\Delta$ is a homology $(d-1)$-sphere follows from the long exact sequence of the pair $(\Delta, \partial\Delta)$ and the Poincaré - Lefschetz duality [Sp, Theorem 6.2.19]. ☐

A simplicial complex Δ is called *pure* if all the facets of Δ have the same dimension. The *dual graph* $\Lambda(\Delta)$ of a pure simplicial complex Δ is the graph whose vertices are the facets of Δ, where two facets are adjacent in $\Lambda(\Delta)$ if they intersect in a face of codimension one. A d-dimensional pure simplicial complex Δ is said to be a *pseudomanifold* if (i) each $(d-1)$-face is in at most two facets, and (ii) the dual graph $\Lambda(\Delta)$ is connected. All connected homology manifolds are pseudomanifolds.

We define the stackedness and the locally stackedness for homology manifolds in the same way as for triangulations of manifolds. Clearly, a stacked homology manifold is locally stacked. Since any stacked homology ball (resp., sphere) is a combinatorial ball (resp., sphere), it follows that a locally stacked homology manifold is a combinatorial manifold, i.e., a PL triangulation of a manifold.

Next, we recall Walkup's class $\mathcal{H}^d$. Let Δ be a connected closed homology manifold and let σ and τ be facets of Δ. We say that a bijection $\psi : \sigma \to \tau$ is *admissible* if $\mathrm{lk}_\Delta(v) \cap \mathrm{lk}_\Delta(\psi(v)) = \{\emptyset\}$ for each vertex $v \in \sigma$. Note that, for the existence of an admissible bijection $\psi : \sigma \to \tau$, σ and τ must be disjoint. For an admissible bijection $\psi : \sigma \to \tau$, let Δ^ψ be the simplicial complex obtained from $\Delta \setminus \{\sigma, \tau\}$ by identifying v and $\psi(v)$ for all $v \in \sigma$. The simplicial complex Δ^ψ is said to be obtained from Δ by a *combinatorial handle addition.*

Definition 2.2 (Walkup's class $\mathcal{H}^d$)**.** Let $d \geq 3$ be an integer. We recursively define the class $\mathcal{H}^d(k)$ as follows.

(a) $\mathcal{H}^d(0)$ is the set of stacked triangulations of the $(d-1)$-sphere.
(b) A simplicial complex Δ is in $\mathcal{H}^d(k+1)$ if it is obtained from a member of $\mathcal{H}^d(k)$ by a combinatorial handle addition.

The Walkup's class $\mathcal{H}^d$ is the union $\mathcal{H}^d = \bigcup_{k \geq 0} \mathcal{H}^d(k)$.

It was proved by Kalai [Ka, Corollary 8.4] that $\mathcal{H}^{d+1}$ is the set of all (locally) stacked triangulations of connected closed d-manifolds for $d \geq 4$. We will extend this result for stacked homology manifolds with and without boundary of dimension ≥ 2.

3. A characterization of stacked triangulated manifolds

All homologies are with coefficients in an arbitrary field $\mathbb{F}$, which is fixed throughout. When the homology manifold Δ means an $\mathbb{F}$-homology manifold then $\widetilde{H}_i(\Delta;\mathbb{F})$ will be denoted by $\widetilde{H}_i(\Delta)$. Similarly, we will denote $\beta(\Delta;\mathbb{F})$ by $\beta(\Delta)$.

We first define an analogue of a combinatorial handle addition for homology manifolds with boundary.

Let Δ be a homology d-manifold with boundary on the vertex set V and let σ and τ be facets of $\partial\Delta$. We say that a bijection $\psi : \sigma \to \tau$ is *admissible* if, for every vertex $v \in \sigma$, $\mathrm{lk}_\Delta(v) \cap \mathrm{lk}_\Delta(\psi(v)) = \{\emptyset\}$. For an admissible bijection $\psi : \sigma \to \tau$, we extend ψ to the map from V to $V \setminus \sigma$ by $\psi(v) = v$ for all $v \notin \sigma$ and write $\psi(\alpha) = \{\psi(v) : v \in \alpha\}$ for any $\alpha \in \Delta$. Define

$$\Delta^{\overline{\psi}} = \{\psi(\alpha) : \alpha \in \Delta\}.$$

Thus $\Delta^{\overline{\psi}}$ is the simplicial complex obtained from Δ by identifying v and $\psi(v)$ for all $v \in \sigma$. If Δ is connected, then we say that $\Delta^{\overline{\psi}}$ is obtained from Δ by a *simplicial handle addition*. If Δ has two connected components Δ_1 and Δ_2 and if $\sigma \in \Delta_1$ and $\tau \in \Delta_2$, then we write $\Delta^{\overline{\psi}} = \Delta_1 \cup_\psi \Delta_2$ and call it a *simplicial connected union* of Δ_1 and Δ_2. Below we give some basic properties of $\Delta^{\overline{\psi}}$.

Lemma 3.1. *Let Δ and Γ be two homology d-balls. If $\Delta \cap \Gamma = \partial\Delta \cap \partial\Gamma = \overline{\alpha}$, where α is a $(d-1)$-simplex, then $\Delta \cup \Gamma$ is a homology d-ball.*

Proof. We use induction on d. The statement is obvious when $d = 1$. Suppose $d > 1$. Since $\Delta \cap \Gamma = \overline{\alpha}$, the Mayer - Vietoris exact sequence says that $\Delta \cup \Gamma$ has a trivial reduced homology. Let v be a vertex of $\Delta \cup \Gamma$. If $v \notin \alpha$ then $\mathrm{lk}_{\Delta\cup\Gamma}(v)$ is equal to either $\mathrm{lk}_\Delta(v)$ or $\mathrm{lk}_\Gamma(v)$ and hence a homology $(d-1)$-sphere or $(d-1)$-ball. If $v \in \alpha$ then $v \in \partial\Delta \cap \partial\Gamma$ and hence $\mathrm{lk}_\Delta(v)$ and $\mathrm{lk}_\Gamma(v)$ are homology $(d-1)$-balls and $\mathrm{lk}_\Delta(v) \cap \mathrm{lk}_\Gamma(v) = \overline{\alpha \setminus \{v\}}$. Since $\mathrm{lk}_{\Delta\cup\Gamma}(v) = \mathrm{lk}_\Delta(v) \cup \mathrm{lk}_\Gamma(v)$, $\mathrm{lk}_{\Delta\cup\Gamma}(v)$ is a homology $(d-1)$-ball by induction hypothesis. The lemma now follows from Lemma 2.1. $\square$

It follows from Lemma 3.1 that the simplicial connected union of two homology d-balls is a homology d-ball.

Lemma 3.2. *Let Δ be a (not necessary connected) homology manifold with boundary of dimension ≥ 2. Let σ and τ be facets of $\partial\Delta$ and $\psi : \sigma \to \tau$ an admissible bijection.*

(i) *$\Delta^{\overline{\psi}}$ is a homology d-manifold with boundary.*
(ii) *$\beta_0(\Delta^{\overline{\psi}})$ is either $\beta_0(\Delta)$ or $\beta_0(\Delta) - 1$. If $\beta_0(\Delta^{\overline{\psi}}) = \beta_0(\Delta)$ then $\beta_1(\Delta^{\overline{\psi}}) = \beta_1(\Delta) + 1$. Otherwise, $\beta_1(\Delta^{\overline{\psi}}) = \beta_1(\Delta)$.*
(iii) *$\Delta^{\overline{\psi}}$ is stacked if and only if Δ is stacked.*

Proof. (i) For every $\alpha \in \Delta^{\overline{\psi}}$ with $\alpha \not\subseteq \tau$, there is a unique face $\gamma \in \Delta$ such that $\alpha = \psi(\gamma)$ and $\mathrm{lk}_{\Delta^{\overline{\psi}}}(\alpha)$ is combinatorially isomorphic to $\mathrm{lk}_\Delta(\gamma)$. Thus, to prove the statement, it is enough to show that, for every $\alpha \subseteq \tau$, $\mathrm{lk}_{\Delta^{\overline{\psi}}}(\alpha)$ is either a homology $(d - \dim(\alpha) - 1)$-sphere or $(d - \dim(\alpha) - 1)$-ball. It is clear that $|\mathrm{lk}_{\Delta^{\overline{\psi}}}(\tau)| \cong S^0$. For a proper face α of τ, a straightforward computation implies

$$\mathrm{lk}_{\Delta^{\overline{\psi}}}(\alpha) = \mathrm{lk}_\Delta(\alpha) \cup_{\psi'} \mathrm{lk}_\Delta(\psi^{-1}(\alpha)),$$

where $\psi' : \psi^{-1}(\tau \setminus \alpha) \to \tau \setminus \alpha$ is the restriction of ψ to $\psi^{-1}(\tau \setminus \alpha)$. By Lemma 3.1, $\mathrm{lk}_{\Delta^{\overline{\psi}}}(\alpha)$ is a homology $(d - \dim(\alpha) - 1)$-ball.

(ii) It is clear that $\beta_0(\Delta^{\overline{\psi}}) = \beta_0(\Delta) - 1$ if σ and τ belong to different connected components and $\beta_0(\Delta^{\overline{\psi}}) = \beta_0(\Delta)$ if σ and τ are in the same connected component. Observe that $\widetilde{H}_i(|\Delta^{\overline{\psi}}|) \cong \widetilde{H}_i(|\Delta^{\overline{\psi}}|, |\tau|) \cong \widetilde{H}_i(|\Delta|, |\sigma| \cup |\tau|)$ for all i. Then the desired statement follows from the following exact sequence of pairs

$$\begin{array}{cccccccc} \cdots & \longrightarrow & \widetilde{H}_1(|\sigma| \cup |\tau|) & \longrightarrow & \widetilde{H}_1(|\Delta|) & \longrightarrow & \widetilde{H}_1(|\Delta|, |\sigma| \cup |\tau|) & \\ & \longrightarrow & \widetilde{H}_0(|\sigma| \cup |\tau|) & \longrightarrow & \widetilde{H}_0(|\Delta|) & \longrightarrow & \widetilde{H}_0(|\Delta|, |\sigma| \cup |\tau|) & \longrightarrow \quad 0. \end{array}$$

(iii) This statement follows from the proof of (i) since it says that the interior faces of $\Delta^{\overline{\psi}}$ are τ and $\psi(\alpha)$ for all interior faces α of Δ. □

The proof of Lemma 3.2 (i) also says that if Δ is connected then $\partial(\Delta^{\overline{\psi}}) = (\partial\Delta)^{\psi}$. Also $|\partial(\Delta_1 \cup_\psi \Delta_2)|$ is a connected sum of $|\partial\Delta_1|$ and $|\partial\Delta_2|$.

Next, we introduce the inverse of the construction of $\Delta^{\overline{\psi}}$, which we call simplicial handle deletions.

Lemma 3.3. *Let B be a homology d-ball with vertex set V, σ an interior $(d-1)$-face of B with $\partial\overline{\sigma} \subset \partial B$. Then $B[V \setminus \sigma]$ contains exactly two connected components.*

Proof. Let $v * \partial B = \partial B \cup \{\{v\} \cup \alpha : \alpha \in \partial B\}$ be the cone of ∂B, where v is a new vertex. It is easy to see that $S = B \cup (v * \partial B)$ is a homology d-sphere. Then

$$\widetilde{H}_0(S[V \setminus \sigma]) \cong \widetilde{H}_{d-1}(S[\sigma \cup \{v\}]) \cong \widetilde{H}_{d-1}(S[\sigma \cup \{v\}], (v * \partial B)[\sigma \cup \{v\}])),$$

where the first isomorphism follows from the Alexander duality [Sp, Theorem 6.2.17] and the second isomorphism follows from the long exact sequence of pairs since $\widetilde{H}_i((v * \partial B)[\sigma \cup \{v\}]) = 0$ for all i. Since $B[V \setminus \sigma] = S[V \setminus \sigma]$ and since

$$\widetilde{H}_{d-1}(S[\sigma \cup \{v\}], (v * \partial B)[\sigma \cup \{v\}])) \cong \widetilde{H}_{d-1}(B[\sigma], (\partial B)[\sigma])) = \widetilde{H}_{d-1}(\overline{\sigma}, \partial\overline{\sigma}) \cong \mathbb{F},$$

$B[V \setminus \sigma]$ has exactly two connected components. □

Recall that any interior $(d-1)$-face σ of a homology d-manifold Δ is contained in exactly two facets since $\mathrm{lk}_\Delta(\sigma)$ has same homologies as S^0.

Lemma 3.4. *Let B and σ be as in Lemma 3.3, C_1 and C_2 the connected components of $B[V \setminus \sigma]$ and let W_1 and W_2 be the vertex set of C_1 and C_2 respectively. Let $B_1 = B[W_1 \cup \sigma]$ and $B_2 = B[W_2 \cup \sigma]$.*

(i) *$B = B_1 \cup B_2$ and $B_1 \cap B_2 = \overline{\sigma}$.*

(ii) *Let $\{x\} \cup \sigma$ and $\{y\} \cup \sigma$ be the facets of B containing σ. Then one of x and y is in B_1 and the other in B_2.*

(iii) *B_1 and B_2 are homology d-balls.*

Proof. (i) It is clear that $B \supset B_1 \cup B_2$ and $B_1 \cap B_2 = \overline{\sigma}$. We prove $B \subset B_1 \cup B_2$. Let α be a facet of B. Then $\alpha \setminus \sigma \in B[V \setminus \sigma]$ is contained in either W_1 or W_2, which implies $\alpha \in B_1 \cup B_2$.

(ii) Since C_1 and C_2 are not empty, there are facets α, γ of B such that $\alpha \in B_1$ and $\gamma \in B_2$. Since B is a pseudomanifold, there is a sequence $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_k = \gamma$ of facets such that $\alpha_{i-1} \cap \alpha_i$ has dimension $d - 1$ for $1 \leq i \leq k$. Let j be a number such that $\alpha_{j-1} \in B_1$ and $\alpha_j \in B_2$. Then $\alpha_{j-1} \cap \alpha_j$ must be σ. Since $\{x\} \cup \sigma$ and $\{y\} \cup \sigma$ are the only facets containing σ, they must be α_{j-1} and α_j.

(iii) We use induction on d. The statement is clear when $d = 1$. Consider the subcomplex B_1. If α is a face in $B_1 \setminus \overline{\sigma}$ then any facet $\gamma \in B$ containing α must intersect W_1 and hence in B_1. If $\alpha \in \overline{\sigma}$, then α is a face of the d-face $\sigma \cup \{x\} \in B_1$. Thus, B_1 is pure. Next, let v be a vertex of B_1. If $v \notin \sigma$ then $\mathrm{lk}_{B_1}(v) = \mathrm{lk}_B(v)$ is a homology $(d-1)$-sphere or a homology $(d-1)$-ball. Suppose $v \in \sigma$. Then $\mathrm{lk}_B(v)$ is a homology $(d-1)$-ball such that $\sigma \setminus \{v\}$ is its interior face. Since $\mathrm{lk}_{\mathrm{lk}_B(v)}(\alpha) = \mathrm{lk}_B(\{v\} \cup \alpha)$ is a homology ball for any $\alpha \in \partial(\overline{\sigma \setminus \{v\}})$, it follows that $\partial(\overline{\sigma \setminus \{v\}}) \subset \partial(\mathrm{lk}_B(v))$. Since x and y are in $\mathrm{lk}_B(v)$, $\mathrm{lk}_B(v)[W_1]$ and $\mathrm{lk}_B(v)[W_2]$ are non-empty. Thus, they are different components of $\mathrm{lk}_B(v)[V \setminus \sigma]$. By induction hypothesis, $\mathrm{lk}_{B_1}(v) = \mathrm{lk}_\Delta(v)[(W_1 \cup \sigma) \setminus \{v\}]$ is a homology $(d-1)$-ball. Thus, $\mathrm{lk}_{B_1}(v)$ is either a homology $(d-1)$-sphere or a homology $(d-1)$-ball for every vertex v of B_1. This implies that B_1 is a homology d-manifold with boundary. Since part (i) and the Mayer - Vietoris exact sequence say $\widetilde{H}_i(B_1) = 0$ for all i, B_1 is a homology d-ball by Lemma 2.1. Similarly, B_2 is a homology d-ball. $\square$

We say that B_i in Lemma 3.4 is the *x-component* (resp. *y-component*) of B with respect to σ if it contains x (resp. y).

Let Δ be a homology d-manifold with boundary. Suppose that Δ has an interior $(d-1)$-face $\sigma = \{z_1, \dots, z_d\}$ with $\partial\overline{\sigma} \subset \partial\Delta$. Let $\{x\} \cup \sigma$ and $\{y\} \cup \sigma$ be the facets of Δ containing σ. Consider

$$R = \{\alpha \in \Delta : \alpha \cap \sigma \neq \emptyset,\ \alpha \not\subseteq \sigma\}.$$

Observe that, for each $\tau \subset \sigma$, $\mathrm{lk}_\Delta(\tau)$ is a homology ball satisfying the assumption of Lemma 3.3 in the sense that $\sigma \setminus \tau$ is an interior face of $\mathrm{lk}_\Delta(\tau)$ with $\partial(\overline{\sigma \setminus \tau}) \subset \partial(\mathrm{lk}_\Delta(\tau))$. Let

$$R_x(k) = \{\alpha \in R : z_k \in \alpha, \alpha \setminus \{z_k\} \text{ is in the } x\text{-component of } \mathrm{lk}_\Delta(z_k) \text{ w.r.t. } \sigma \setminus \{z_k\}\}$$

and define $R_y(k)$ similarly. Let

$$X = \bigcup_{k=1}^{d} R_x(k) \text{ and } Y = \bigcup_{k=1}^{d} R_y(k).$$

Note that $R = X \cup Y$.

Lemma 3.5. *If $R_x(k)$, $R_y(k)$, X and Y are as above then $X \cap Y = \emptyset$. Also, $\{\alpha \in X : z_k \in \alpha\} = R_x(k)$ and $\{\alpha \in Y : z_k \in \alpha\} = R_y(k)$ for $1 \leq k \leq d$.*

Proof. To prove the first result, what we must prove is that $R_x(k) \cap R_y(\ell) = \emptyset$ for all $k \neq \ell$. Suppose contrary that $\alpha \in R_s(k) \cap R_y(\ell)$ for some $k \neq \ell$. Then $\alpha \setminus \{z_k, z_\ell\}$ is in the x-component and the y-component of $\mathrm{lk}_\Delta(\{z_k, z_\ell\})$ with respect to $\sigma \setminus \{z_k, z_\ell\}$ and hence $\alpha \subseteq \sigma$, a contradiction since $\alpha \in R$.

Let $\alpha \in X$ with $z_k \in \alpha$. Then $\alpha \in R_x(\ell)$ for some ℓ. If $\ell = k$ then $\alpha \in R_x(k)$. Otherwise, $\alpha \setminus \{z_k, z_\ell\}$ and x are in the same component of $\mathrm{lk}_\Delta(\{z_k, z_\ell\})$. Since $\mathrm{lk}_\Delta(z_k) \supset \mathrm{lk}_\Delta(\{z_k, z_\ell\})$, we have $\alpha \in R_x(k)$. This proves that $\{\alpha \in X : z_k \in \alpha\} = R_x(k)$. Similarly, $\{\alpha \in Y : z_k \in \alpha\} = R_y(k)$. $\square$

Definition 3.6. Let Δ be a homology d-manifold with boundary and let $\sigma = \{z_1, \dots, z_d\}$ be an interior $(d-1)$-face of Δ with $\partial\overline{\sigma} \subset \partial\Delta$. Let R, $R_x(k)$, $R_y(k)$, X and Y be as above. Let $z_1^+, \dots, z_d^+$ be new vertices and $\sigma^+ = \{z_1^+, \dots, z_d^+\}$. For $\alpha = \alpha' \cup \{z_{i_1}, \dots, z_{i_\ell}\} \in X$ with $\alpha' \cap \sigma = \emptyset$, define $\alpha^+ = \alpha' \cup \{z_{i_1}^+, \dots, z_{i_\ell}^+\}$. Consider the simplicial complex

$$\widetilde{\Delta}^\sigma = \{\alpha \in \Delta : \alpha \notin X\} \cup \{\alpha^+ : \alpha \in X\} \cup \overline{\sigma^+}.$$

We say that $\widetilde{\Delta}^\sigma$ is obtained from Δ by a *simplicial handle deletion* over σ.

Intuitively, $\widetilde{\Delta}^\sigma$ is a simplicial complex obtained from Δ by cutting it along the face σ. Note that this construction is a simplified version of the construction in [BD1, Lemma 3.3]. Also, a similar construction for manifolds without boundary was considered by Walkup [Wa].

Theorem 3.7. *Let $\widetilde{\Delta}^\sigma$ be obtained from a homology d-manifold with boundary Δ by a* simplicial handle deletion *over σ. Then*

(i) *$\widetilde{\Delta}^\sigma$ is a homology d-manifold with boundary, and*
(ii) *$\Delta = (\widetilde{\Delta}^\sigma)^{\overline{\psi}}$, where $\psi : \sigma^+ \to \sigma$ is the bijection given by $\psi(z_i^+) = z_i$ for all i.*

Proof. The second statement is straightforward if $\widetilde{\Delta}^\sigma$ is a homology manifold. So, we prove (i). For simplicity, we write $\widetilde{\Delta} = \widetilde{\Delta}^\sigma$. Let V be the vertex set of Δ.

We prove that each vertex link of $\widetilde{\Delta}$ is either a homology $(d-1)$-sphere or a $(d-1)$-ball. Suppose $v \notin \sigma^+ \cup \sigma$. Define the map $\varphi : \Delta \to \widetilde{\Delta}$ by $\varphi(\alpha) = \alpha$ if $\alpha \notin X$ and $\varphi(\alpha) = \alpha^+$ if $\alpha \in X$. Then φ gives a bijection between $\Delta \setminus \overline{\sigma}$ and $\widetilde{\Delta} \setminus (\overline{\sigma^+} \cup \overline{\sigma})$, in particular, gives a bijection between $\{\alpha : v \in \alpha \in \Delta\}$ and $\{\alpha : v \in \alpha \in \widetilde{\Delta}\}$. Thus $\mathrm{lk}_{\widetilde{\Delta}}(v)$ is combinatorially isomorphic to $\mathrm{lk}_\Delta(v)$, which implies the desired property. Suppose $v = z_k^+$ for some k. Then

$$\mathrm{lk}_{\widetilde{\Delta}}(v) = \mathrm{lk}_{\widetilde{\Delta}}(z_k^+) = \overline{(\sigma \setminus \{z_k\})^+} \cup \{(\alpha \setminus \{z_k\})^+ : z_k \in \alpha \in X\}.$$

On the other hand, the x-component of $\mathrm{lk}_\Delta(z_k)$ is

$$\overline{\sigma \setminus \{z_k\}} \cup \{(\alpha \setminus \{z_k\}) : z_k \in \alpha \in R_x(k)\}.$$

By Lemma 3.5, they are combinatorially isomorphic. This proves that $\mathrm{lk}_{\widetilde{\Delta}}(v)$ is a homology $(d-1)$-ball. Finally, suppose $v = z_k$ for some k. Since $X \cap Y = \emptyset$,

$$\begin{aligned}\mathrm{lk}_{\widetilde{\Delta}}(v) &= \overline{\sigma \setminus \{v\}} \cup \{\alpha \setminus \{v\} : v \in \alpha \in R_y(k) \setminus X\} \\ &= \overline{\sigma \setminus \{z_k\}} \cup \{\alpha \setminus \{z_k\} : \alpha \in R_y(k)\}\end{aligned}$$

is the y-component of $\mathrm{lk}_\Delta(v)$ w.r.t. $\sigma \setminus \{v\}$. Thus $\mathrm{lk}_{\widetilde{\Delta}}(v)$ is a homology $(d-1)$-ball.

Finally, $\widetilde{\Delta}$ has a non-empty boundary since $\sigma \in \partial\widetilde{\Delta}$. □

Now, we define an analogue of Walkup's class for manifolds with boundary.

Definition 3.8. Let $d \geq 2$ be an integer. We recursively define $\overline{\mathcal{H}^d}(k)$ as follows.

(a) $\overline{\mathcal{H}^d}(0)$ is the set of stacked triangulations of d-balls.
(b) Δ is a member of $\overline{\mathcal{H}^d}(k+1)$ if it is obtained from a member of $\overline{\mathcal{H}^d}(k)$ by a simplicial handle addition.

Let $\overline{\mathcal{H}^d} = \bigcup_{k\geq 0} \overline{\mathcal{H}^d}(k)$.

Note that every stacked triangulation of the d-ball is obtained from a d-simplex by taking a simplicial connected union with a d-simplex repeatedly. See [DS, Lemma 2.1]. Also, if $\Delta \in \overline{\mathcal{H}^d}$ then $\partial\Delta \in \mathcal{H}^d$.

Lemma 3.9. *If $\Delta \in \overline{\mathcal{H}^d}(k)$ and $\Gamma \in \overline{\mathcal{H}^d}(\ell)$ then their simplicial connected union belongs to $\overline{\mathcal{H}^d}(k+\ell)$.*

Proof. We may assume $k \leq \ell$. We use induction on $k+\ell$. If $k+\ell = 0$ then the assertion follows from Lemma 3.2(iii). Suppose $k+\ell > 0$. Then $\Gamma = \Sigma^{\overline{\varphi}}$ for some $\Sigma \in \overline{\mathcal{H}^d}(\ell-1)$ and for some admissible bijection φ between facets of $\partial\Sigma$. Let ψ be a bijection from a facet of $\partial\Delta$ to a facet of $\partial\Gamma$. Then $\Delta \cup_\psi \Gamma$ is $(\Delta \cup_\psi \Sigma)^{\varphi}$ (by an appropriate identification of the vertices). By induction hypothesis, we have $\Delta \cup_\psi \Sigma \in \overline{\mathcal{H}^d}(k+\ell-1)$ and hence $\Delta \cup_\psi \Gamma \in \overline{\mathcal{H}^d}(k+\ell)$. □

Remark 3.10. A similar result for $\mathcal{H}^d$ was proved by Walkup [Wa, Proposition 4.4].

Theorem 3.11. *Let Δ be a connected homology manifold Δ with boundary of dimension $d \geq 2$. Then Δ is stacked if and only if $\Delta \in \overline{\mathcal{H}^d}$.*

Proof. The 'if part' follows from part (iii) of Lemma 3.2.

The 'only if part' is obvious if Δ has one facet. Suppose that Δ has more than one facet. Then Δ has an interior $(d-1)$-face σ. Since Δ is stacked, it has no interior faces of dimension $\leq d-2$. Thus we have $\partial\overline{\sigma} \subset \partial\Delta$. By Lemma 3.2 and Theorem 3.7, Δ is a simplicial connected union of two connected stacked homology manifolds or is obtained from a connected stacked homology manifold having a smaller first Betti number by a simplicial handle addition. Then the assertion follows by double induction on the number of facets and the first Betti number. □

Since, the boundary of a member of $\overline{\mathcal{H}^d}$ is a member of $\mathcal{H}^d$ for $d \geq 3$, we obtain the following corollary.

Corollary 3.12. *Let Δ be a connected closed homology manifold of dimension $d \geq 2$. Then Δ is stacked if and only if $\Delta \in \mathcal{H}^{d+1}$.*

We say that a connected closed $\mathbb{F}$-homology d-manifold Δ is *orientable* if $\widetilde{H}_d(\Delta;\mathbb{F}) \cong \mathbb{F}$. Novik and Swartz [NS, Theorem 5.2] and Bagchi [Ba, Theorem 1.14] gave the following interesting characterization of the stackedness.

Proposition 3.13 (Novik - Swartz)**.** *Let Δ be a connected closed orientable homology manifold of dimension $d \geq 3$. Then*

$$f_1(\Delta) - (d+1)f_0(\Delta) + \binom{d+2}{2} \geq \binom{d+2}{2}\beta_1(\Delta).$$

Further, if $d \geq 4$ then $f_1(\Delta) - (d+1)f_0(\Delta) + \binom{d+2}{2} = \binom{d+2}{2}\beta_1(\Delta)$ if and only if $\Delta \in \mathcal{H}^{d+1}$.

Proposition 3.14 (Bagchi)**.** *Let Δ be a connected closed homology 3-manifold. Then $f_1(\Delta) - 4f_0(\Delta) + 10 \geq 10\beta_1(\Delta)$. Equality holds here if and only if Δ is stacked.*

As a consequence of Corollary 3.12 and Proposition 3.14 we have

Corollary 3.15. *Let Δ be a connected closed homology 3-manifold. Then $f_1(\Delta) - 4f_0(\Delta) + 10 = 10\beta_1(\Delta)$ if and only if $\Delta \in \mathcal{H}^4$.*

This shows that Proposition 3.13 is true for $d = 3$ also. This is a question ([NS, Problem 5.3]) posed by Novik and Swartz. Observe that, Corollary 3.15 is valid without the assumption of any orientability. We think that the following conjecture, which is a special case of [BD4, Conjecture 1.6] posed by Bagchi and the first author, is plausible.

Conjecture 3.16. Proposition 3.13 holds for any connected closed homology manifold.

Remark 3.17. It is known that the topological type of a member of $\mathcal{H}^{d+1}$ is one of the following: (i) the d-sphere S^d (ii) connected sums of sphere product $S^{d-1} \times S^1$ (iii) connected sums of twisted sphere product $S^{d-1} \underline{\times} S^1$. See [LSS, Section 3]. Thus Corollary 3.12 also gives a restriction to the topological types of stacked triangulations of manifolds.

4. Tight triangulations and stackedness

In this section, we study stackedness of tight triangulations. As in the previous section, we fix a field $\mathbb{F}$ and denote $\widetilde{H}(\Delta;\mathbb{F})$ and $\beta_i(\Delta;\mathbb{F})$ by $\widetilde{H}(\Delta)$ and $\beta(\Delta)$. Also, we simply say that a simplicial complex is tight if it is $\mathbb{F}$-tight. For a simplicial complex Δ with vertex set V, a subset $\sigma \subset V$ of $k+1$ elements is called a *missing k-face* of Δ if $\sigma \notin \Delta$ and all proper subsets of σ are faces of Δ. If σ is a missing k-face of Δ, then we have $\widetilde{H}_{k-1}(\Delta[\sigma]) \cong \mathbb{F}$. The following lemma follows from the definition of tightness.

Lemma 4.1. *Let Δ be an tight simplicial complex on the vertex set V. Then*

(i) *for all subsets $U \subset W$ of V, the natural map $\widetilde{H}_i(\Delta[U]) \to \widetilde{H}_i(\Delta[W])$ induced by the inclusion is injective, and*

(ii) *if $\beta_{k-1}(\Delta) = 0$ then Δ has no missing k-faces. In particular, if Δ is connected, then Δ is neighborly.*

For a simplicial complex Δ, we identify its 1-*skeleton* $\mathrm{Skel}_1(\Delta) = \{\sigma \in \Delta : \dim(\sigma) \leq 1\}$ with the simple graph whose vertex set is the set of the vertices of Δ and whose edge set is the set of the edges (1-simplices) in Δ. We say that a simple graph G is *chordal* if it has no induced cycle of length ≥ 4. The following result is due to Kalai [Ka, Theorem 8.5].

Proposition 4.2 (Kalai)**.** *Let Δ be a homology $(d-1)$-sphere with $d \geq 3$. Then Δ is stacked if and only if the 1-skeleton of Δ is chordal and Δ has no missing k-faces for $1 < k < d-1$.*

Let Δ be a connected closed homology manifold of dimension $d \geq 3$. Recall that Δ is said to be tight-neighborly if $\binom{f_0(\Delta)-d-1}{2} = \binom{d+2}{2}\beta_1(\Delta;\mathbb{Z}/2\mathbb{Z})$. Since $\binom{f_0}{2} - (d+1)f_0 + \binom{d+2}{2} = \binom{f_0-d-1}{2}$, by Propositions 3.13 and 3.14, Δ is tight-neighborly if and only if Δ is stacked and neighborly. Note that these propositions also say that, for a tight-neighborly triangulation Δ, $\beta_1(\Delta;\mathbb{Z}/2\mathbb{Z}) = \beta_1(\Delta;\mathbb{F})$ for any field $\mathbb{F}$. Here we prove the following.

Theorem 4.3. *Let Δ be an tight connected closed homology manifold of dimension $d \geq 4$ such that $\beta_i(\Delta) = 0$ for $1 < i < d-1$. Then Δ is locally stacked.*

Proof. Let v be a vertex of Δ. We prove that $\mathrm{lk}_\Delta(v)$ is stacked.

We first claim that any induced subcomplex of $\mathrm{lk}_\Delta(v)$ cannot be a 1-dimensional simplicial complex which forms a cycle. Suppose contrary that $\mathrm{lk}_\Delta(v)[W]$ is a cycle for some W. Let $C = \mathrm{lk}_\Delta(v)[W]$ and $v * C = C \cup \{\{v\} \cup \sigma : \sigma \in C\}$. Then we have $\Delta[W \cup \{v\}] = \Delta[W] \cup (v*C)$ and $\Delta[W] \cap (v*C) = C$. Consider the Mayer - Vietoris exact sequence

$$\widetilde{H}_2(\Delta[W \cup \{v\}]) \longrightarrow \widetilde{H}_1(C) \longrightarrow \widetilde{H}_1(\Delta[W]) \oplus \widetilde{H}_1(v * C) \xrightarrow{\varphi} \widetilde{H}_1(\Delta[W \cup \{v\}]).$$

Since Δ is tight and $\beta_2(\Delta) = 0$, we have $\widetilde{H}_2(\Delta[W \cup \{v\}]) = 0$. Then since $\widetilde{H}_1(C) \neq 0$, the map φ has a non-trivial kernel. However, since $\widetilde{H}_1(v*C) = 0$, this contradicts the tightness of Δ as it implies that φ is injective by Lemma 4.1(i). Hence any induced subcomplex of $\mathrm{lk}_\Delta(v)$ cannot be a cycle.

Now we prove the statement. By Lemma 4.1(ii), Δ has no missing k-faces for $2 < k < d$. This implies that $\mathrm{lk}_\Delta(v)$ has no missing k-faces for $2 < k < d-1$. Also, $\mathrm{lk}_\Delta(v)$ has no missing 2-faces since if it has a missing 2-face σ then $\mathrm{lk}_\Delta(v)[\sigma]$ is a cycle of length 3. Similarly, the 1-skeleton of $\mathrm{lk}_\Delta(v)$ is a chordal graph since if it has an induced cycle of length ≥ 4 with the vertex set W, then $\mathrm{lk}_\Delta(v)[W]$ is a cycle. Thus, by Proposition 4.2, $\mathrm{lk}_\Delta(v)$ is stacked. □

From this theorem and all the known results, we have the following.

Corollary 4.4. *Let Δ be a connected closed homology manifold of dimension $d \geq 4$. Then the following are equivalent.*

(i) Δ *is tight-neighborly.*
(ii) Δ *is a neighborly member of* $\mathcal{H}^{d+1}$.
(iii) Δ *is neighborly and stacked.*
(iv) Δ *is neighborly and locally stacked.*
(v) Δ *is tight and* $\beta_i(\Delta) = 0$ *for* $1 < i < d-1$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from Proposition 3.13. The equivalence (ii) $\Leftrightarrow$ (iii) follows from Corollary 3.12. The equivalence (ii) $\Leftrightarrow$ (iv) follows from Kalai's result [Ka, Corollary 8.4].

Now, (v) $\Rightarrow$ (iv) follows from Theorem 4.3. Since $\Delta \in \mathcal{H}^{d+1}$ implies $\beta_i(\Delta) = 0$ for $1 < i < d-1$, (ii) & (iv) $\Rightarrow$ (v) follows from Effenberger's result [Ef, Theorem 3.2]. This completes the proof. □

From the equivalence of (i) and (v) in Corollary 4.4 it follows that tight triangulations of connected sums of S^{d-1}-bundles over S^1 are tight-neighborly for $d \geq 4$. This answers a question asked by Effenberger [Ef, Question 4.5].

It would be natural to ask following.

Question 4.5. Is every tight triangulation of a connected closed 3-manifold locally stacked?

Acknowledgements: The authors thank Bhaskar Bagchi for many useful comments on a preliminary version of this paper. They also thank Eran Nevo for letting them know that Lemma 3.3 follows from the Alexander duality. The first author was partially supported by DIICCSRTE, Australia and DST, India, under the Australia-India Strategic Research Fund (project AISRF06660) and by the UGC Centre for Advanced Studies. The second author was partially supported by JSPS KAKENHI 26400054.

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