THE NUMBERS OF EDGES OF 5-POLYTOPES WITH A GIVEN NUMBER OF VERTICES

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ABSTRACT. A basic combinatorial invariant of a convex polytope P is its f-vector $f(P) = (f_0, f_1, \dots, f_{\dim P-1})$, where f_i is the number of i-dimensional faces of P. Steinitz characterized all possible f-vectors of 3-polytopes and Grünbaum characterized the pairs given by the first two entries of the f-vectors of 4-polytopes. In this paper, we characterize the pairs given by the first two entries of the f-vectors of 5-polytopes. The same result was also proved by Pineda-Villavicencio, Ugon and Yost independently.

1. INTRODUCTION

The study of f-vectors of convex polytopes is one of the central research topic in convex geometry. We call a d-dimensional convex polytope a *d-polytope*. For a convex polytope (or a polyhedral complex) P, we write $f_i(P)$ for the number of i-dimensional faces of P. The *f-vector* of a d-polytope P is the vector $f(P) = (f_0(P), f_1(P), \dots, f_{d-1}(P))$. In 1906, Steinitz characterized all possible f-vectors of 3-polytopes (see [Gr, §10.3]). While a characterization of f-vectors of 4-polytopes is a big open problem in convex geometry, for any $0 \le i < j \le 3$, the following set was characterized by Grünbaum [Gr], Barnette [Ba] and Barnette–Reay [BR] (see also [BL, Theorem 3.9])

$$\big\{(f_i(P), f_j(P)) : P \text{ is a 4-polytope}\big\}.$$

Moreover, Sjöberg and Ziegler [SZ] recently characterize all possible values of the pairs (f_0, f_{03}) of flag face numbers of 4-polytopes. In this paper, we characterize all possible (f_0, f_1) pairs of 5-polytopes.

Let

$$\mathcal{E}^d = \{(f_0(P), f_1(P)) : \ P \text{ is a } d\text{-polytope}\}.$$

The set $\mathcal{E}^3$ was determined by Steinitz in 1906 who shows that

$$\mathcal{E}^3 = \{(v, e) : \tfrac{3}{2}v \le e \le 3v - 6\}.$$

Note that, by Euler's relation, this actually determines all possible f-vectors of 3-polytopes. In higher dimensions, it is easy to see that any d-polytope P satisfies

$$(1) \qquad \frac{d}{2} f_0(P) \le f_1(P) \le \binom{f_0(P)}{2}.$$

Indeed, the first inequality follows since $f_1(P)$ equals to $\frac{1}{2}$ times the sum of degrees of the vertices of P and since each vertex of P has degree $\ge d$. Grünbaum [Gr,

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§10.4] proved that the inequality (1) characterizes $\mathcal{E}^4$, with four exceptions. More precisely, he proved the following statement.

Theorem 1.1 (Grünbaum)**.**

$$\mathcal{E}^4 = \left\{(v,e) : 2v \le e \le \binom{v}{2}\right\} \setminus \big\{(6,12),(7,14),(8,17),(10,20)\big\}.$$

In dimension 5, the situation is more complicated. The set $\mathcal{E}^5$ is close to the set of integer points satisfying (1), but there are not only a finite list of exceptions but also an infinite family of exceptions. Indeed, our main result is the following.

Theorem 1.2. *Let* $L = \{(v, \lfloor \frac{5}{2}v + 1\rfloor) : v \ge 7\}$ *and* $G = \{(8,20),(9,25),(13,35)\}$. *Then*

$$\mathcal{E}^5 = \left\{(v,e) : \frac{5}{2}v \le e \le \binom{v}{2}\right\} \setminus (L \cup G).$$

Here $\lfloor a \rfloor$ denotes the integer part of a rational number a. Note that it is not hard to see $(8,20) \notin \mathcal{E}^5$ since a 5-polytope P with $f_1(P) = \frac{5}{2}f_0(P)$ must be a simple polytope. Also, $(9,25) \notin \mathcal{E}^5$ was proved in [PUY1] recently.

The following table illustrates the shape of $\mathcal{E}^5$.

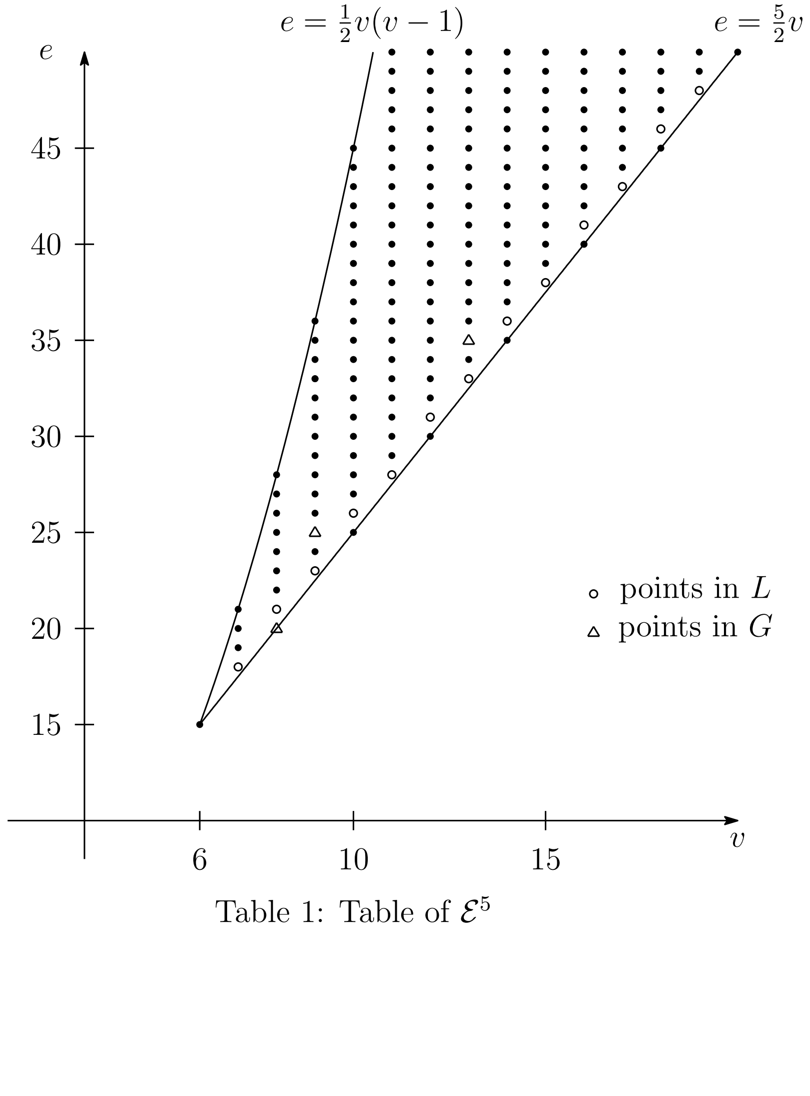


Table 1: Table of $\mathcal{E}^5$

In the table, black dots represent points in $\mathcal{E}^5$, white circles and triangles represent points in L and G respectively. For example, on the line $v = 9$, $(9, 23) \in L$ is presented by a white circle, $(9, 25) \in G$ is presented by a triangle, and the possible numbers of edges are $24, 26, 27, \dots, 36$.

Theorem 1.2 was also independently proved by Pineda-Villavicencio, Ugon and Yost [PUY2] by a different method.

It would be interesting to determine $\mathcal{E}^d$ for $d \geq 6$, and more generally to characterize the set $\{(f_i(P), f_j(P)) : P \text{ is a } d\text{-polytope}\}$ for any $0 \leq i < j < d$. About the latter problem, Sjöberg and Ziegler [SZ] recently study the case when $i = 0$ and $j = d - 1$.

2. SUFFICIENCY

In this section, we prove the sufficiency part of Theorem 1.2. If a polytope Q is the pyramid over a polytope P, then we have

$$f_0(Q) = f_0(P) + 1 \text{ and } f_1(Q) = f_0(P) + f_1(P).$$

This simple fact and Theorem 1.1 prove the next lemma.

Lemma 2.1.

$$\mathcal{E}^5 \supset \left\{(v, e) : 3v - 3 \leq e \leq \binom{v}{2}\right\} \setminus \big\{(7, 18), (8, 21), (9, 25), (11, 30)\big\}.$$

Let P be a d-polytope. The *degree* $\deg v$ of a vertex v of P is the number of edges of P that contain v. We say that a vertex v is *simple* if $\deg v = d$. Let $V(P)$ be the vertex set of P.

Lemma 2.2. *If P is a 5-polytope such that $f_1(P) \leq 3f_0(P) - 1$, then P has a simple vertex.*

Proof. Observe $\deg v \geq 5$ for any $v \in V(P)$. Since $\frac{1}{2}\sum_{v \in V(P)} \deg v = f_1(P) = 3f_0(P) - 1$, there must exist a vertex $v \in V(P)$ of degree < 6. □

Let

$$X = \big\{(v, e) : \tfrac{5}{2}v \leq e \leq \tbinom{v}{2}\big\} \setminus (L \cup G)$$

be the right-hand side of Theorem 1.2, and let

$$X_k = \{(v, e) \in X : v = k\}$$

for $k \in \mathbb{Z}$. We want to prove $\mathcal{E}^5 \supset X_k$ for all $k \geq 6$. To prove this, we use truncations. For a 5-polytope P and its vertex $v \in V(P)$, we write $\mathrm{tr}(P, v)$ for a polytope obtained from P by truncating the vertex v. If v is simple, then

$$f_0(\mathrm{tr}(P, v)) = f_0(P) + 4 \text{ and } f_1(\mathrm{tr}(P, v)) = f_1(P) + 10.$$

Lemma 2.3. *For $k \geq 6$ with $k \notin \{8, 9, 13\}$, if $\mathcal{E}^5 \supset X_k$ then $\mathcal{E}^5 \supset X_{k+4}$.*

Proof. Since $k \notin \{8, 9, 13\}$,

$$X_k = \big\{(k, e) : \tfrac{5}{2}k \leq e \leq \tbinom{k}{2}\big\} \setminus \big\{\big(k, \lfloor \tfrac{5}{2}k + 1 \rfloor\big)\big\}.$$

By Lemma 2.2, for any 5-polytope P with $f_0(P) = k$ and $f_1(P) \le 3k-1$, we can make a 5-polytope Q with

$$f_0(Q) = k+4 \text{ and } f_1(Q) = f_1(P)+10$$

by truncating a simple vertex from P. Since $\mathcal{E}^5 \supset X_k$, this implies

$$\begin{aligned}\mathcal{E}^5 &\supset \left\{(k+4, e+10) : \frac{5}{2}k \le e \le 3k-1\right\} \setminus \left\{\left(k+4, \left\lfloor \frac{5}{2}k + 11 \right\rfloor\right)\right\} \\ &= \left\{\big((k+4), e'\big) : \frac{5}{2}(k+4) \le e' \le 3(k+4)-3\right\} \setminus \left\{\left(k+4, \left\lfloor \frac{5}{2}(k+4)+1 \right\rfloor\right)\right\}.\end{aligned}$$

The above inclusion and Lemma 2.1 prove the desired statement. □

Now, we prove the main result of this section. For a convex polytope P, we write P^* for its dual polytope. In the rest of the paper, if a face of a convex polytope is a simplex, then we call it a *simplex face*. A face which is not a simplex is called a *non-simplex face*.

Theorem 2.4. $\mathcal{E}^5 \supset X$.

Proof. By Lemma 2.3, it is enough to show that

$$\mathcal{E}^5 \supset X_6 \cup X_7 \cup X_8 \cup X_9 \cup X_{12} \cup X_{13} \cup X_{17}. \tag{2}$$

Let $\varphi(v) = 3v-3$. By Lemma 2.1, $(v,e) \in \mathcal{E}^5$ if $\varphi(v) \le e \le \binom{v}{2}$ and $(v,e) \notin \{(7,18),(8,21),(9,25),(11,30)\}$. Observe $\varphi(6) = 15, \varphi(7) = 18, \varphi(8) = 21, \varphi(9) = 24, \varphi(12) = 33, \varphi(13) = 36, \varphi(17) = 48$. Then, to prove (2), what we must prove is

$$\mathcal{E}^5 \supset \{(12,30),(12,32),(13,34),(17,44),(17,45),(17,46),(17,47)\}. \tag{3}$$

(See also Table 1.) Note that this observation says $\mathcal{E}^5 \supset X_k$ for $k \le 9$.

Let C be the cyclic 5-polytope with 7 vertices. Then $f(C) = (7,21,34,30,12)$ (see [Br, §18]). Hence $f_0(C^*) = 12$ and $f_1(C^*) = 30$, and therefore $(12,30) \in \mathcal{E}^5$. Let C' be the polytope obtained from C^* by truncating its vertex. Note that every vertex of C^* is simple. Since a truncation of a simple vertex creates a simplex facet, C' contains a simplex facet F. Let C'' be the polytope obtained from C' by adding a pyramid over F. Then

$$f_0(C'') = f_0(C')+1 = f_0(C^*)+5 = 17$$

and

$$f_1(C'') = f_1(C')+5 = f_1(C^*)+15 = 45.$$

Hence $(17,45) \in \mathcal{E}^5$. We already see $(8,22),(9,24) \in \mathcal{E}^5$. Then, using truncations of simple vertices and Lemma 2.2, we have $(12,32),(13,34),(17,44) \in \mathcal{E}^5$. Also, since $(13,36),(13,37) \in \mathcal{E}^5$, by the same argument we have $(17,46),(17,47) \in \mathcal{E}^5$. These complete the proof of the theorem. □

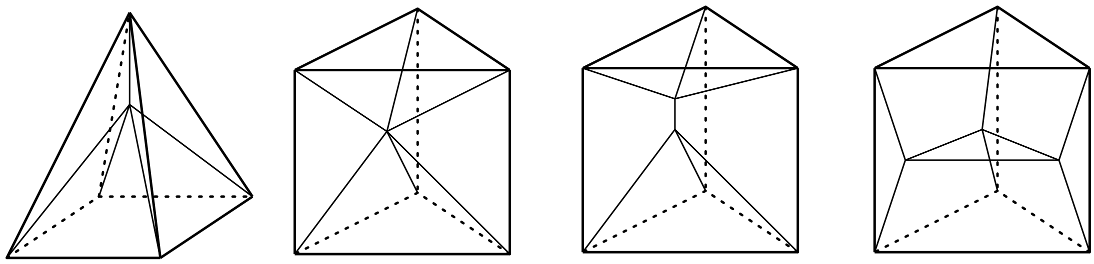
3. Necessity

In this section, we prove the necessity part of Theorem 1.2. We first show that any element in L is not contained in $\mathcal{E}^5$. We introduce some lemmas which we need. The following fact appears in [Gr, §6.1] (see also [Zi, Problem 6.8]).

Lemma 3.1. *There are exactly four combinatorially different* 4*-polytopes with* 6 *facets. They are*

(P_A) *Pyramid over a square pyramid;*
(P_B) *Pyramid over a triangular prism;*
(P_C) *A polytope obtained from a* 4*-simplex by truncating its vertex;*
(P_D) *Product of two triangles.*

Here are Schlegel diagrams and a list of facets of P_A, P_B, P_C and P_D.



P_A P_B P_C P_D

Type	Facets
P_A	two square pyramids, four tetrahedra
P_B	three square pyramids, one triangular prism, two tetrahedra
P_C	four triangular prisms, two tetrahedra
P_D	six triangular prisms

Recall that a convex polytope P is said to be *simplicial* if all its proper faces are simplices. A *simplicial k-sphere* is a simplicial complex whose geometric realization is homeomorphic to the k-sphere. The boundary complex of a simplicial d-polytope is a simplicial $(d-1)$-sphere. The next statement easily follows from the Lower Bound Theorem (see [Ka]) and the Upper Bound Theorem (see [St, Corollary II.3.5]) for simplicial spheres.

Lemma 3.2. *Let* Δ *be a simplicial* 3*-sphere.*

(i) $f_3(\Delta) \neq 6, 7, 10$.
(ii) *If* $f_3(\Delta) = 9$*, then* Δ *is neighbourly, that is, every pair of vertices of* Δ *are connected by an edge.*

Proof. By the Lower Bound Theorem and the Upper Bound Theorem, we have

- if $f_0(\Delta) = 5$, then $5 \leq f_3(\Delta) \leq 5$;
- if $f_0(\Delta) = 6$, then $8 \leq f_3(\Delta) \leq 9$;
- if $f_0(\Delta) \geq 7$, then $11 \leq f_3(\Delta)$.

These clearly imply (i). The statement (ii) follows from the fact that if the number of facets of a simplicial $(d-1)$-sphere equals to the bound in the Upper Bound Theorem, then it must be neighbourly (see e.g. the proof of [Br, Theorem 18.1]). □

We now prove that any element in L is not contained in $\mathcal{E}^5$.

Proposition 3.3. *If P is a 5-polytope, then $f_1(P) \neq \lfloor \frac{5}{2} f_0(P) + 1 \rfloor$.*

Proof. Suppose to the contrary that $f_1(P) = \lfloor \frac{5}{2} f_0(P) + 1 \rfloor$. We first consider the case when $f_0(P)$ is odd. Then, since $\sum_{v \in V(P)} \deg v = 2f_1(P) = 5f_0(P) + 1$, P has one vertex having degree 6 and all other vertices have degree 5. Then P^* has one facet F with $f_3(F) = 6$ and all other facets of P^* are simplices. However this implies that the 4-polytope F must be simplicial, which contradicts Lemma 3.2(i).

Next, we consider the case when $f_0(P)$ is even. In this case, $\sum_{v \in V(P)} \deg v = 2f_1(P) = 5f_0(P) + 2$, so one of the following two cases occurs:

(a) P^* has one facet F with $f_3(P) = 7$ and all other facets of P^* are simplices;
(b) P^* has two facets F and G with $f_3(F) = f_3(G) = 6$ and all other facets of P^* are simplices.

Since there are no simplicial 4-polytope with 7 facets by Lemma 2.2(i), the case (a) cannot occur. Also, if the case (b) occurs, then F and G can have at most one non-simplex facet. However, Lemma 3.1 says that any 4-polytope with 6 facets have at least two non-simplex facets. □

Next, we show that any element of G is not contained in $\mathcal{E}^5$. Let $\phi(v,d) = \frac{1}{2}dv + \frac{1}{2}(v-d-1)(2d-v)$. The following result was proved by Pineda-Villavicencio, Ugon and Yost [PUY1, Theorems 6 and 19].

Theorem 3.4. *Let P be a d-polytope.*

(i) *If $f_0(P) \leq 2d$, then $f_1(P) \geq \phi(f_0(P), d)$.*
(ii) *If $d \geq 4$, then $(f_0(P), f_1(P)) \neq (d+4, \phi(d+4, d) + 1)$.*

By considering the special case when $d = 5$ of the above theorem, we obtain the following.

Corollary 3.5. $(8, 20), (9, 25) \notin \mathcal{E}^5$.

By Proposition 3.3 and Corollary 3.5, to prove Theorem 1.2, we only need to prove $(13, 35) \notin \mathcal{E}^5$. We will prove this in the rest of this paper.

Let P be a 5-polytope. For faces $F_1, F_2, \ldots, F_k$ of P, we write $\langle F_1, \ldots, F_k \rangle$ for the polyhedral complex generated by $F_1, \ldots, F_k$. Let $\{G_1, \ldots, G_l\}$ be a subset of the set of facets of P. Then any 3-face of $\Gamma = \langle G_1, \ldots, G_l \rangle$ is contained in at most two facets of Γ. We write

$$\partial\Gamma = \langle H \in \Gamma : \ H \text{ is a 3-face of } \Gamma \text{ contained in exactly one facet of } \Gamma \rangle.$$

We often use the following trivial observation: If $\{G_1, \ldots, G_l\}$ is the set of non-simplex facets of P, then $\partial\langle G_1, \ldots, G_l \rangle$ is a simplicial complex.

We say that a d-polytope P is *almost simplicial* if all facets of P except for one facet are simplices. (We consider that simplicial polytopes are not almost simplicial.)

The next lemma can be checked by using a complete list of 4-polytopes with at most 8 vertices (see [FMM]), but we write its proof for completeness.

Lemma 3.6. *Let P be a 4-polytope.*

(i) *Suppose that P is almost simplicial and $f_3(P) = 7$. Then P is the pyramid over a triangular bipyramid.*
(ii) *Suppose that P is almost simplicial and $f_3(P) = 8$. Then P does not contain a triangular bipyramid as a facet.*
(iii) *Suppose that $f_3(P) = 7$ and P has exactly two non-simplex facets F and G. Then none of F and G are square pyramids.*

Proof. (i) Let F be the unique non-simplex facet of P. Clearly, F is simplicial and $f_2(F) \leq f_3(P) - 1 = 6$ since, for each 2-face of F, there is a unique 3-face of P that contains it other than F. Since a 3-simplex and a triangular bipyramid are the only simplicial 3-polytopes having at most 6 facets, F is a triangular bipyramid. Then, since P has 7 facets, P must be the pyramid over F.

(ii) Let F be the unique non-simplex facet of P. If F is a triangular bipyramid, then by subdividing F into two tetrahedra without introducing edges, one obtains a simplicial 3-sphere Δ with 9 facets. Since F is a triangular bipyramid, there are two vertices u and v of F such that u and v are not connected by an edge in F. These vertices are not connected by an edge in Δ by the construction of Δ, which contradicts Lemma 3.2(ii) saying that Δ must be neighbourly.

(iii) Suppose to the contrary that F is a square pyramid. We claim that G is also a square pyramid. Indeed, since $\partial\langle F, G\rangle$ is a simplicial complex, G contains exactly one non-simplex facet, and this facet must be a square and equals to $F \cap G$. Also,

$$f_2(G) \leq f_3(P) - 1 = 6.$$

Let Δ be a simplicial 2-sphere obtained from G by subdividing the square $F \cap G$ into two triangles. Then $f_2(\Delta) \leq 7$, but since the number of 2-faces of a simplicial 2-sphere is even, $f_2(\Delta) = 6$ and therefore $f_2(G) = 5$. This forces that G is a square pyramid.

Let $F = \mathrm{conv}(v_1, a, b, c, d)$ and $G = \mathrm{conv}(v_2, a, b, c, d)$, where $\mathrm{conv}(v_1, \ldots, v_k)$ denotes the convex hull of points $v_1, \ldots, v_k$. Note that $\mathrm{conv}(a, b, c, d)$ is a square. We assume that $\mathrm{conv}(a, c)$ and $\mathrm{conv}(b, d)$ are non-edges of P. By subdividing each F and G into two tetrahedra by adding an edge $\mathrm{conv}(a, c)$, we can make a simplicial 3-sphere Γ with $f_3(\Gamma) = f_3(P) + 2 = 9$. By the construction of Γ, $\mathrm{conv}(b, d)$ is not an edge of Γ, but this contradicts Lemma 3.2(ii). □

We also recall some known results on h-vectors of simplicial balls and their boundaries. For a simplicial complex Δ of dimension $d - 1$, its *h-vector* $h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_d(\Delta))$ is defined by

$$h_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1}(\Delta)$$

where $f_{-1}(\Delta) = 1$. A *simplicial d-ball* is a simplicial complex whose geometric carrier is homeomorphic to a d-dimensional ball. The following facts are known. See [St, Chapter II and Problem 12].

Lemma 3.7. *Let Δ be a simplicial d-ball and $(h_0, h_1, \dots, h_{d+1})$ its h-vector. Then*

(i) $h_0 = 1$ *and* $h_{d+1} = 0$.
(ii) $h_0 + \cdots + h_{d+1} = f_d(\Delta)$.
(iii) $h_i \geq 0$ *for all* i.
(iv) *The h-vector of the boundary complex of Δ is*

$$(h_0, h_0 + h_1 - h_d, h_0 + h_1 + h_2 - h_d - h_{d-1}, \cdots).$$

We now complete the proof of Theorem 1.2.

Theorem 3.8. $(13, 35) \notin \mathcal{E}^5$.

Proof. Suppose to the contrary that there is a 5-polytope P such that $f_0(P) = 13$ and $f_1(P) = 35$. Let $v_1, \dots, v_{13}$ be the vertices of P with $\deg v_1 \geq \cdots \geq \deg v_{13}$ and let $D = (\deg v_1, \deg v_2, \dots, \deg v_{13})$. Since

$$\sum_{k=1}^{13} \deg v_k = 2f_1(P) = 70$$

and $\deg v_k \geq 5$ for each k, D must be one of the following.

(1) $D = (10, 5, \dots, 5)$;
(2) $D = (9, 6, 5, \dots, 5)$;
(3) $D = (8, 7, 5, \dots, 5)$;
(4) $D = (8, 6, 6, 5, \dots, 5)$;
(5) $D = (7, 7, 6, 5, \dots, 5)$;
(6) $D = (7, 6, 6, 6, 5, \dots, 5)$;
(7) $D = (6, 6, 6, 6, 6, 5, \dots, 5)$.

Below we will show a contradiction for each case.

(1) Suppose $D = (10, 5, \dots, 5)$. Then P^* has a 4-face F with $f_3(F) = 10$. This F must be a simplicial 4-polytope since all other facets of P^* are simplices, which contradicts Lemma 3.2(i) saying that there are no simplicial 4-polytopes with 10 facets.

(2) Suppose $D = (9, 6, 5, \dots, 5)$. P^* has only two non-simplex 4-faces F and G. These 4-faces can have at most one non-simplex 3-face. By the assumption on D, F or G must have 6 facets. This contradicts Lemma 3.1 saying that any 4-polytope with 6 facets has at least two non-simplex facets.

(3) Suppose $D = (8, 7, 5, \dots, 5)$. Let F and G be the 4-faces of P^* with $f_3(F) = 8$ and $f_3(G) = 7$. Then G is not simplicial by Lemma 3.2(i) and therefore F is also not simplicial. Hence F and G are almost simplicial and $F \cap G$ is a 3-polytope which is not a simplex. Since $f_3(F) = 8$ and $f_3(G) = 7$, this contradicts Lemma 3.6(i) and (ii).

(4) Suppose $D = (8, 6, 6, 5, \dots, 5)$. Let F, G and G' be 4-faces of P^* with $f_3(F) = 8$ and $f_3(G) = f_3(G') = 6$. Since $\partial\langle F, G, G'\rangle$ is a simplicial complex, the polytopes F, G and G' have at most two non-simplex 3-faces. By Lemma 3.1, G and G' must

be the polytope P_A, and F must have 2 square pyramids as its 3-faces. Thus, the 4-polytope F has 6 tetrahedra and 2 square pyramids as its facets. This implies

$$f_0(F^*) = f_3(F) = 8 \quad \text{and} \quad f_1(F^*) = f_2(F) = \frac{1}{2}(5 \times 2 + 4 \times 6) = 17.$$

However, this contradicts Theorem 1.1 saying that $(8, 17) \notin \mathcal{E}^4$.

(5) Suppose $D = (7, 7, 6, 5, \ldots, 5)$. Let F, F' and G be 4-faces of P^* with $f_3(F) = f_3(F') = 7$ and $F(G) = 6$. Since all other 4-faces of P^* are simplices, F, F' and G have at most two non-simplex 3-faces. Then, by Lemma 3.1, G must be the polytope P_A, and F and F' have a square pyramid as its facets. By Lemma 3.6(i), F and F' are not almost simplicial. Hence F and F' have exactly two non-simplex facets, but this contradicts Lemma 3.6(iii).

(6) Suppose $D = (7, 6, 6, 6, 5, \ldots, 5)$. Let F, G, G' and G'' be 4-facets of P^* with $f_3(F) = 7$ and $f_3(G) = f_3(G') = f_3(G'') = 6$. Since all other 4-faces of P^* are simplices, each of F, G, G' and G'' can have at most three non-simplex 3-faces. By Lemma 3.1, G, G' and G'' must be the polytope P_A and have exactly two non-simplex 3-faces. Since $\partial\langle F, G, G', G''\rangle$ is a simplicial complex, one of the following situations must occur:

(a) F is a simplicial polytope;
(b) F has exactly two square pyramids as its facets and all other facets are simplices.

However, (a) cannot occur by Lemma 3.2(i) and (b) cannot occur by Lemma 3.6(iii).

(7) Suppose $D = (6, 6, 6, 6, 6, 5, \ldots, 5)$. Let $F_1, \ldots, F_5$ be the 4-faces of P^* with $f_3(F_1) = \cdots = f_3(F_5) = 6$. Observing that each of $F_1, \ldots, F_5$ must be one of P_A, P_B, P_C and P_D in Lemma 3.1. It is not hard to see that one of the following situations must occur;

(a) All the $F_1, \ldots, F_5$ are P_A;
(b) All the $F_1, \ldots, F_5$ are P_C;
(c) F_1 and F_2 are P_B. F_3, F_4 and F_5 are P_A.

We first show that (a) cannot occur. We may assume that $F_i \cap F_{i+1}$ is a square pyramid for $i = 1, 2, \ldots, 5$, where $F_6 = F_0$. Since P_A has only one square as its 2-faces, $K = F_1 \cap F_2 \cap \cdots \cap F_5$ must be this square, and the polyhedral complex generated by the facets of P^* that do not contain K is a shellable simplicial 4-ball by a line shelling [Zi, §8.2]. Let B be this ball. Clearly,

$$B = \langle G : G \text{ is a simplex facet of } P^*\rangle.$$

Hence B has $13 - 5 = 8$ facets. Also, since each $F_i \cap F_{i+1}$ is a pyramid over the square K, the faces $F_1, \ldots, F_5$ can be written as

$$F_1 = \operatorname{conv}(v_1, v_2, u_1, u_2, u_3, u_4),$$

$$F_2 = \operatorname{conv}(v_2, v_3, u_1, u_2, u_3, u_4),$$

$$\vdots$$

$$F_5 = \operatorname{conv}(v_5, v_1, u_1, u_2, u_3, u_4)$$

with $K = \mathrm{conv}(u_1, \ldots, u_4)$. Then it follows that $\Gamma = \partial\langle F_1, \ldots, F_5\rangle$ is the join of the 5-cycle and the 4-cycle, and its h-vector is

$$(1, 5, 8, 5, 1)$$

since its entries coincide with the coefficients of the polynomial $(1+3t+t^2)(1+2t+t^2)$. Let $h(B) = (1, h_1, h_2, h_3, h_4, 0)$. Lemmas 3.7 and 3.8 say

$$1 + h_1 - h_4 = 5,\ 1 + h_1 + h_2 - h_3 - h_4 = 8 \text{ and } 1 + h_1 + h_2 + h_3 + h_4 = 8.$$

Then it is easy to see $h(B) = (1, 4, 3, 0, 0, 0)$. Let $G_1, \ldots, G_8$ be a shelling of B. Let

$$R_j = \{v \in V(G_j) : \mathrm{conv}(V(G_j) \setminus \{v\}) \in \langle G_1, \ldots, G_{j-1}\rangle\}$$

and $S_j = V(G_j) \setminus R_j$, where $V(G_j)$ is the vertex set of G_j. Then

$$h_i = h_i(B) = |\{j : |R_j| = i\}|$$

for all i (see [Zi, §8.3]). Since $h_3 = h_4 = 0$, we have $|S_j| = 5 - |R_j| \geq 3$ for all j. By the definition of a shelling, $\mathrm{conv}(S_8)$ is a missing face of ∂B, that is, $\mathrm{conv}(S_8)$ is not a face of ∂B but any its proper face is a face of ∂B. Thus ∂B has a missing face of dimension ≥ 2. However, the join of two cycles of length ≥ 4 does not have any missing face of dimension ≥ 2, and $\partial B = \partial\langle F_1, \ldots, F_5\rangle$ is the join of the 5-cycle and the 4-cycle, a contradiction.

We next prove that (b) cannot occur. Observe that P_C has 4 non-simplex facets. Since $\partial\langle F_1, \ldots, F_5\rangle$ is a simplicial complex, each $F_i \cap F_j$ must be a triangular prism. Then it is easy to see that we can write

$$\begin{aligned}
F_1 &= \mathrm{conv}(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4),\\
F_2 &= \mathrm{conv}(x_1, x_2, x_3, x_5, y_1, y_2, y_3, y_5),\\
F_3 &= \mathrm{conv}(x_1, x_2, x_4, x_5, y_1, y_2, y_4, y_5),\\
F_4 &= \mathrm{conv}(x_1, x_3, x_4, x_5, y_1, y_3, y_4, y_5),\\
F_5 &= \mathrm{conv}(x_2, x_3, x_4, x_5, y_2, y_3, y_4, y_5),
\end{aligned}$$

where each $\mathrm{conv}(x_i, x_j, x_k, y_i, y_j, y_k)$ is a triangular prism with triangles $\mathrm{conv}(x_i, x_j, x_k)$ and $\mathrm{conv}(y_i, y_j, y_k)$ (we assume that each $\mathrm{conv}(x_k, y_k)$ is an edge of P^*). Using this formula, one conclude that

$$\begin{aligned}
&\partial\langle F_1, F_2, \ldots, F_5\rangle\\
&= \langle \mathrm{conv}(S) : S \subset \{x_1, \ldots, x_5\},\ |S| = 4\rangle \cup \langle \mathrm{conv}(S) : S \subset \{y_1, \ldots, y_5\},\ |S| = 4\rangle
\end{aligned}$$

is the disjoint union of two copies of the boundary of a 4-simplex.

If $\mathrm{conv}(x_1, \ldots, x_5)$ is not a face of P^*, then, for each $S \subset \{x_1, \ldots, x_5\}$ with $|S| = 4$, there is a unique 4-face $G_S \notin \{F_1, \ldots, F_5\}$ of P^* that contains $\mathrm{conv}(S)$. This implies that, since P^* has only 8 facets other than $F_1, \ldots, F_5$, either $\mathrm{conv}(x_1, \ldots, x_5)$ or $\mathrm{conv}(y_1, \ldots, y_5)$ must be a face of P^* (otherwise P^* has at least 10 facets other than $F_1, \ldots, F_5$). We assume that $\mathrm{conv}(x_1, \ldots, x_5)$ is a face of P^*. Let $G_1, \ldots, G_8$ be the simplex 4-faces of P^* and assume $G_1 = \mathrm{conv}(x_1, \ldots, x_5)$. Then $\Gamma = \langle G_2, \ldots, G_8\rangle$ is a pseudomanifold with

$$\partial\Gamma = \langle \mathrm{conv}(S) : S \subset \{y_1, \ldots, y_5\},\ |S| = 4\rangle.$$

Let n_i be the number of interior vertices of Γ. By the Lower Bound Theorem for pseudomanifolds with boundary [Fo] (see also [Ta, Theorem 1.2]), Γ must have at least $5 + 4n_i - 4$ facets. Since Γ only has 7 facets, we have $n_i \leq 1$. However, this implies that Γ is either the 4-simplex or the cone over the boundary of the 4-simplex, contradicting the fact that Γ has 7 facets.

We finally prove that (C) cannot occur. Since $\partial\langle F_1, \dots, F_5\rangle$ is a simplicial complex, $F_1 \cap F_2$ must be a triangular prism and $F_i \cap F_j$ is a square pyramid for all $i \in \{1, 2\}$ and $j \in \{3, 4, 5\}$. It is not hard to see that $F_1, \dots, F_5$ can be written as

$$\begin{aligned}
F_1 &= \mathrm{conv}(x, y, z, x', y', z', v_1),\\
F_2 &= \mathrm{conv}(x, y, z, x', y', z', v_2),\\
F_3 &= \mathrm{conv}(x, y, x', y', v_1, v_2),\\
F_4 &= \mathrm{conv}(x, z, x', z', v_1, v_2),\\
F_5 &= \mathrm{conv}(y, z, y', z', v_1, v_2),
\end{aligned}$$

where $\mathrm{conv}(x, y, z, x', y', z')$ is a triangular pyramid with triangles $\mathrm{conv}(x, y, z)$ and $\mathrm{conv}(x', y', z')$ (we assume that $\mathrm{conv}(x, x'), \mathrm{conv}(y, y')$ and $\mathrm{conv}(z, z')$ are edges). A routine computation shows

$$\begin{aligned}
\partial\langle F_1, F_2, \dots, F_5\rangle =&\langle \mathrm{conv}(S) : S \subset \{v_1, v_2, x, y, z\},\ |S| = 4\rangle\\
&\cup \langle \mathrm{conv}(S) : S \subset \{v_1, v_2, x', y', z'\},\ |S| = 4\rangle
\end{aligned}$$

is the union of two copies of the boundary of a 4-simplex intersections in the edge $\mathrm{conv}(v_1, v_2)$. Then the exactly same argument as in the case (b) works, namely, one case show that either $\mathrm{conv}(v_1, v_2, x, y, z)$ or $\mathrm{conv}(v_1, v_2, x', y', z')$ must be a face of P^* and conclude a contradiction by the Lower Bound Theorem for pseudomanifolds with boundary. □

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