BETTI NUMBERS OF CHORDAL GRAPHS AND f-VECTORS OF SIMPLICIAL COMPLEXES

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Abstract. Let G be a chordal graph and $I(G)$ its edge ideal. Let $\beta(I(G)) = (\beta_0, \beta_1, \dots, \beta_p)$ denote the Betti sequence of $I(G)$, where β_i stands for the ith total Betti number of $I(G)$ and where p is the projective dimension of $I(G)$. It will be shown that there exists a simplicial complex Δ of dimension p whose f-vector $f(\Delta) = (f_0, f_1, \dots, f_p)$ coincides with $\beta(I(G))$.

Introduction

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K with $\deg x_i = 1$ for any i. The *Betti sequence* of a homogeneous ideal $I \subset S$ is the sequence

$$\beta(I) = (\beta_0(I), \beta_1(I), \dots, \beta_p(I)),$$

where each $\beta_i(I)$ stands for the ith total Betti number of I and where $p = \operatorname{proj}\dim(I)$ is the projective dimension of I. One has $\sum_{i=-1}^{p}(-1)^i\beta_i(I) = 0$ with $\beta_{-1}(I) = 1$.

Let Δ be a simplicial complex and

$$f(\Delta) = (f_0, f_1, \dots, f_{d-1})$$

its *f-vector*, where each $f_i = f_i(\Delta)$ stands for the number of faces of Δ of dimension i and where $d-1$ is the dimension of Δ. Recall that Δ is *acyclic* (over K) if its reduced homology group $\tilde{H}_i(\Delta; K)$ with coefficients K vanishes for all i. Thus in particular if Δ is acyclic, then its f-vector satisfies $\sum_{i=-1}^{d-1}(-1)^i f_i = 0$ with $f_{-1} = 1$.

Peeva and Velasco [20] succeeded in proving that, given an acyclic simplicial complex Δ, there exists a monomial ideal I whose Betti sequence $\beta(I)$ coincides with the f-vector $f(\Delta)$. In general, the converse is, however, false. Let $n = 6$ and $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_1x_6)$. Then $\dim S/I = 3$, $\operatorname{depth} S/I = 2$ and $p = 4$. One has $\beta(I) = (6, 9, 6, 2)$. If a simplicial complex Δ possesses 2 faces of dimension 3, then Δ possesses at least 7 faces of dimension 2. It then follows that there exists *no* simplicial complex Δ of dimension 3 with $(6, 9, 6, 2)$ its f-vector.

On the other hand, in Example 1.8, one can find a Cohen–Macaulay monomial ideal I, i.e., S/I is a Cohen–Macaulay ring, whose Betti sequence is the f-vector of a simplicial complex, but *not* the f-vector of an acyclic simplicial complex.

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It is natural to ask which monomial ideals I enjoy the property that there exists a simplicial complex (or acyclic simplicial complex) Δ whose f-vector coincides with the Betti sequence of I. The purpose of the present paper is to establish the research project on finding a natural class $\mathcal{C}$ of monomial ideals such that, for each ideal I belonging to $\mathcal{C}$, the Betti sequence $\beta(I)$ is the f-vector of a simplicial (or an acyclic simplicial) complex.

First, in Section 1, we summarize several answers, which are easily or directly obtained from well-known facts. The topics discussed will include monomial ideals with small projective dimensions, cellular resolutions, componentwise linear ideals and pure resolutions.

Now, Section 2 is the highlight of this paper. Let G be a finite graph on the vertex set V and $E(G)$ the edge set of G. We write $S = K[\{x : x \in V\}]$ for the polynomial ring in $|V|$ variables over a field K with $\deg x = 1$ for any $x \in V$. The edge ideal of G is the ideal $I(G)$ of S generated by those monomials xy with $\{x, y\} \in E(G)$. Recall that a finite graph G is chordal if each cycle of G of length > 3 has a chord. Theorem 2.1 guarantees that, for an arbitrary chordal graph G, there exists a simplicial complex Δ whose f-vector coincides with $\beta(I(G))$. The recursive-type formula due to Hà and Van Tuyl [12] will be indispensable to achieve the proof of Theorem 2.1.

Finally, in Section 3, we study Gorenstein monomial ideals. It follows that the Betti sequence of a Gorenstein monomial ideal I with $\operatorname{proj}\dim(I) \leq 3$ is the f-vector of an acyclic simplicial complex. On the other hand, we can characterize the possible Betti numbers of Gorenstein monomial ideals I with $\operatorname{proj}\dim(I) = 3$. Moreover, it will be proved that, given integers $m \geq 4$ and $p \geq 3$, there exists a Gorenstein monomial ideal I of $K[x_1, \ldots, x_n]$, where n is enough large, with $\beta_0(I) = m$ and $\operatorname{proj}\dim(I) = p$ if and only if $m \geq p + 1$ with $m \neq p + 2$.

1. Betti sequences and acyclic simplicial complexes

The present section is a summary of several answers, which are easily or directly obtained from well-known facts, for the problem of finding a natural class $\mathcal{C}$ of monomial ideals such that, for each ideal I belonging to $\mathcal{C}$, the Betti sequence $\beta(I)$ is the f-vector of a simplicial (or an acyclic simplicial) complex.

First, recall a combinatorial characterization of f-vectors of acyclic simplicial complexes due to Gil Kalai [17].

Lemma 1.1 (Kalai). *A vector $f = (f_0, f_1, \ldots, f_{d-1})$ of positive integers is the f-vector of an acyclic simplicial complex of dimensional $d-1$ if and only if there exists a simplicial complex Δ' of dimension $d-2$ with $f(\Delta') = (f'_0, f'_1, \ldots, f'_{d-2})$ such that $f_i = f'_i + f'_{i-1}$ for all i, where $f'_{-1} = 1$ and $f'_{d-1} = 0$.*

(1.1) Monomial ideals with small projective dimensions

Let $I \subset S$ be a monomial ideal with $\operatorname{proj}\dim(I) \leq 2$ and $\beta(I) = (n, \beta_1, \beta_2)$. One has $1 - n + \beta_1 - \beta_2 = 0$. It follows from the Taylor resolution of I that there exists an integer $c \geq 0$ such that $\beta_1 = \binom{n}{2} - c$. Thus $\beta(I) = (n, \binom{n}{2} - c, \binom{n-1}{2} - c)$. Since $(n-1, \binom{n-1}{2} - c)$ is the f-vector of a simplicial complex, Lemma 1.1 says that $\beta(I) = (n, \binom{n}{2} - c, \binom{n-1}{2} - c)$ is the f-vector of an acyclic simplicial complex.

Theorem 1.2. *Let $I \subset S$ be a monomial ideal with* $\operatorname{proj}\dim(I) \leq 2$*. Then $\beta(I)$ is the f-vector of an acyclic simplicial complex.*

(1.2) Cellular resolutions

The cellular resolution was introduced by Bayer and Sturmfels [2]. Let $I \subset S$ be a monomial ideal and $\mathbf{F}_\bullet$ a $\mathbb{Z}^n$-graded free resolution of S/I. The complex $\mathbf{F}_\bullet \otimes_S S/(x_1 - 1, \ldots, x_n - 1)$ of K-vector spaces is called the *frame* of $\mathbf{F}_\bullet$. We say that $\mathbf{F}_\bullet$ is supported by a CW-complex Δ if its frame is equal to the augmented oriented chain complex of Δ. If a free resolution is supported by a CW-complex Δ, then Δ must be acyclic ([2, Proposition 1.2]). Thus if a minimal free resolution is supported by a simplicial complex, then its Betti sequence must be the f-vector of an acyclic simplicial complex.

A monomial ideal $I \subset S$ is said to be *generic* if, for all pairs of generators $u = x_1^{a_1} \ldots x_n^{a_n}$ and $v = x_1^{b_1} \ldots x_n^{b_n}$ of I, one has $a_k \neq b_k$ or $a_k = b_k = 0$ for all k. It was proved by Bayer, Peeva and Sturmfels [3] that a generic monomial ideal has a minimal free resolution which is supported by a simplicial complex.

Theorem 1.3. *Let I be a generic monomial ideal. Then $\beta(I)$ is the f-vector of an acyclic simplicial complex.*

We say that a CW-complex Δ satisfies the *intersection property* if the intersection of two faces of Δ is again a face of Δ. For example, all simplicial complexes as well as all polyhedral complexes satisfy the intersection property. Björner and Kalai [5] proved that if Δ is an acyclic regular CW-complex satisfying the intersection property, then the f-vector of Δ is the f-vector of an acyclic simplicial complex.

Theorem 1.4. *Suppose that the minimal free resolution of a monomial ideal $I \subset S$ is supported by a regular CW-complex satisfying the intersection property. Then $\beta(I)$ is the f-vector of an acyclic simplicial complex.*

Velasco [23] studied minimal free resolutions which are not supported by a CW-complex by means of the nearly scarf ideal introduced in [20]. Let Ω be a simplicial complex with the vertex set $[n] = \{1, 2, \ldots, n\}$ which is not the boundary of a simplex. The *nearly scarf ideal* J_Ω of Ω is the monomial ideal of the polynomial ring $K[x_\sigma : \sigma \in \Omega \setminus \{\emptyset\}]$ generated by $\{\prod_{\sigma \in \Omega,\ v \notin \sigma} x_\sigma : v \in [n]\}$. It is known [20] that the graded Betti numbers of J_Ω is given by

$$\beta_i(J_\Omega) = f_i(\Omega) + \dim_K \tilde{H}_{i-1}(\Omega; K), \qquad i \geq 0.$$

On the other hand, Björner–Kalai Theorem ([4]), which gives a characterization of the (f,β)-pairs of simplicial complexes, guarantees that, for an arbitrary simplicial complex Δ with $f(\Delta)=(f_0,f_1,\dots,f_{d-2})$, the vector $(f'_0,\dots,f'_{d-1})$ defined by setting $f'_i=f_i+\dim_K \tilde{H}_{i-1}(\Delta;K)$ is the f-vector of an acyclic simplicial complex.

Theorem 1.5. *Let J_Ω be the nearly scarf ideal of Ω. Then $\beta(J_\Omega)$ is the f-vector of an acyclic simplicial complex.*

(1.3) Componentwise linear ideals

One of the most famous classes of monomial ideals for which the formula of graded Betti numbers is known is the class of stable ideals. Recall that a monomial ideal $I\subset S$ is *stable* if, for all monomials $u\in I$ and for all $1\le i<m(u)$, one has $ux_i/x_{m(u)}\in I$, where $m(u)$ is the maximal integer k such that x_k divides u. Let I be a stable ideal and $G(I)$ the minimal set of monomial generators of I. Write $m_k(I)$ for the number of monomials $u\in G(I)$ with $m(u)=k$. Eliahou and Kervaire [10] proved that

$$\beta_i(I)=\sum_{k=i+1}^{n} m_k(I)\binom{k-1}{i}$$

for all $i\ge 0$.

A homogeneous ideal $I\subset S$ is said to have a *k-linear resolution* if $\beta_{i,i+j}(I)=0$ whenever $j\ne k$. A homogeneous ideal $I\subset S$ is said to be *componentwise linear* ([14]) if, for all integers $k\ge 0$, the ideal $I_{\langle k\rangle}$ which is generated by the homogeneous polynomials of degree k belonging to I has a k-linear resolution. A *quasi-forest* is a simplicial complex Δ whose Stanley–Reisner ideal I_Δ has a 2-linear resolution. It is known (Fröberg [11]) that a quasi-forest is the clique complex of a chordal graph.

Theorem 1.6. *Let $\beta=(\beta_0,\beta_1,\dots,\beta_p)$ with $p\le n-1$ be a sequence of integers. The following conditions are equivalent:*

(i) *There exists a componentwise linear ideal $I\subset S$ with* $\operatorname{proj\,dim}(I)=p$ *such that $\beta(I)=\beta$;*
(ii) *There exists a stable ideal $I\subset S$ with* $\operatorname{proj\,dim}(I)=p$ *such that $\beta(I)=\beta$;*
(iii) *There exists a sequence $c_1,\dots,c_{p+1}$ of positive integers with $c_1=1$ such that $\beta_i=\sum_{k=1}^{p+1}c_k\binom{k-1}{i}$ for all $i\ge 0$;*
(iv) *There exists an acyclic quasi-forest Δ of dimension p such that $\beta=f(\Delta)$.*

Proof. First, (i) $\Leftrightarrow$ (ii) is known ([8, Lemma 1.4]). Second, (ii) $\Rightarrow$ (iii) follows from Eliahou–Kervaire formula and the fact that if I is a stable ideal and $m_k(I)\ne 0$ for some $k>0$, then $m_\ell(I)\ne 0$ for all $1\le \ell<k$ ([15, Lemma 1.3]). Third, to prove (iii) $\Rightarrow$ (ii), we introduce the monomial ideal I generated by

$$\bigcup_{i=1}^{p+1}\left\{(x_1^{c_2}\cdots x_{i-2}^{c_{i-1}})x_{i-1}^{c_i+1-k}x_i^{c_{i+1}+k} : k=1,\dots,c_i\right\},$$

where $c_{p+2} = 0$. It follows that I is stable and $(m_1(I), \ldots, m_{p+1}(I)) = (c_1, \ldots, c_{p+1})$.

Finally, (iii) $\Leftrightarrow$ (iv) will be shown. It is known [16] that $f = (f_0, f_1, \ldots, f_{p-1})$ is the f-vector of a quasi-forest of dimension $p-1$ if and only if there exists a sequence of positive integers $b_1, \ldots, b_p$ such that $f_{i-1} = \sum_{k=1}^{p} b_k \binom{k-1}{i-1}$ for all $i \geq 1$. If Δ is a quasi-forest, then it follows from [13, Theorem 7.1] that its algebraic shifted complex Σ is again a quasi-forest. If Δ is acyclic then Σ must be a cone ([17]). However, if a quasi-forest Σ is a cone, then it must be a cone of a quasi-forest. These facts guarantee that $f = (f_0, f_1, \ldots, f_p)$ is the f-vector of an acyclic quasi-forest of dimension p if and only if f is the f-vector of a cone of a quasi-forest of dimension $p-1$. The latter condition is equivalent to saying that there exists a sequence of positive integers $b_1, \ldots, b_p$ such that $f_{i-1} = \sum_{k=1}^{p} \{b_k \binom{k-1}{i-1} + b_k \binom{k-1}{i-2}\} = \sum_{k=1}^{p} b_k \binom{k}{i-1}$ for all $i \geq 2$ and $f_0 = 1 + \sum_{k=1}^{p} b_k$. Set $c_1 = 1$ and $c_k = b_{k-1}$ for $k = 2, 3, \ldots, p+1$. Then the sequence $c_1, \ldots, c_{p+1}$ satisfies the conditions of (iii), as desired. ☐

(1.4) Pure resolutions

We discuss the question whether Betti sequences of monomial ideals with pure resolutions are f-vectors of simplicial complexes. We say that a homogeneous ideal $I \subset S$ has a *pure resolution* if its minimal free resolution is of the form

$$0 \longrightarrow S(-c_p)^{\beta_p} \longrightarrow S(-c_{p-1})^{\beta_{p-1}} \longrightarrow \cdots \longrightarrow S(-c_0)^{\beta_0} \longrightarrow I \longrightarrow 0.$$

Let $v > d \geq 1$ and $C(v, d)$ the cyclic polytope [21, p. 59] of dimension d with v vertices. Since $C(v, d)$ is a simplicial polytope, its boundary $\partial C(v, d)$ defines a simplicial complex $\Delta(C(v, d))$, called the boundary complex of $C(v, d)$. It is known [22, Proposition 3.1] that, when d is even, the Stanley–Reisner ideal $I_{\Delta(C(v,d))}$ ([21, p. 53]) of $\Delta(C(v, d))$ has a pure resolution.

Example 1.7. Let $v = 7$ and $d = 2$. Then the Betti sequence of $I_{\Delta(C(7,2))}$ is $(14, 35, 35, 14, 1)$. In particular $(14, 35, 35, 14, 1)$ is the Betti sequence arising from a pure resolution. However, it turns out that $(14, 35, 35, 14, 1)$ cannot be the Betti sequence arising from a linear resolution.

Example 1.8. In [7] it is shown that there exists a simplicial complex Δ such that (i) I_Δ has a pure, but not a linear resolution; (ii) the Betti sequence of I_Δ is $\beta(I_\Delta) = (14, 21, 14, 6)$; (iii) the Stanley–Reisner ring $K[\Delta] = S/I_\Delta$ ([21, p. 53]) is Cohen–Macaulay. Now, Kruskal–Katona theorem [21, p. 55] says that $(14, 21, 14, 6)$ is the f-vector of a simplicial complex. However, by using Lemma 1.1 it turns out that $(14, 21, 14, 6)$ cannot be the f-vector of an acyclic simplicial complex.

Theorem 1.9. *If d is even, then the Betti sequence of $I_{\Delta(C(v,d))}$ is the f-vector of a simplicial complex.*

Proof. Let $d = 2d'$ and $\beta(I_{\Delta(C(v,d))}) = (\beta_0, \ldots, \beta_{v-2d'-1})$. It follows from [22] that

$$(1) \qquad \beta_i = \binom{v-d'-1}{d'+i+1}\binom{d'+i}{d'} + \binom{v-d'-1}{i}\binom{v-d'-i-2}{d'}$$

for $i < v - 2d' - 1$ and $\beta_{v-2d'-1} = 1$.

Let $v = d + 1$. Then the Betti sequence of $I_{\Delta(C(v,d))}$ is (1), which is the f-vector of a 0-simplex. Let $v \geq d + 2$. Our proof will be done by using induction on d'.

Let $d' = 1$. Then $\Delta(C(v, 2))$ is a cycle with v vertices. We show that, by using induction on v, the Betti sequence $\beta(I_{\Delta(C(v,2))})$ is the f-vector of a simplicial complex. When $v = 4$, the Betti sequence of $I_{\Delta(C(v,2))}$ is $(2, 1)$, which is the f-vector of a 1-simplex. Let $v > 4$ and suppose that there exists a simplicial complex $\Gamma(v-1)$ such that $f(\Gamma(v-1)) = \beta(I_{\Delta(C(v-1,2))})$.

Let x_0 be a new vertex and write $\{x_0\} * \Gamma(v-1)$ for the cone over $\Gamma(v-1)$ with the vertex x_0. In other words,

$$\{x_0\} * \Gamma(v-1) = \{\{x_0\} \cup F \,:\, F \in \Gamma(v-1)\} \cup \Gamma(v-1).$$

By using the formula (1) it follows easily that

$$\beta_i(I_{\Delta(C(v,2))}) = \begin{cases} f_0(\{x_0\} * \Gamma(v-1)) + v - 3, & i = 0, \\ f_i(\{x_0\} * \Gamma(v-1)) + \binom{v-2}{i+1}, & 1 \leq i \leq v-5, \\ f_{v-4}(\{x_0\} * \Gamma(v-1)) + v - 3, & i = v-4, \\ 1, & i = v-3. \end{cases}$$

Let $x_1, \ldots, x_{v-3}$ be new vertices and Γ' the simplicial complex consisting of all subsets of $\{x_0, x_1, \ldots, x_{v-3}\}$. We then introduce the simplicial complex $\Gamma(v)$ by setting

$$\Gamma(v) = (\{x_0\} * \Gamma(v-1)) \cup (\Gamma' \setminus \{\{x_0, x_1, \ldots, x_{v-3}\}, \{x_1, \ldots, x_{v-3}\}\}).$$

Since $f_{v-3}(\{x_0\} * \Gamma(v-1)) = f_{v-4}(\Gamma(v-1)) = 1$, one has $\beta_i(I_{\Delta(C(v,2))}) = f_i(\Gamma(v))$ for all i, as desired.

Next, let $d' > 1$. Again, we show that, by using induction on v, the Betti sequence $\beta(I_{\Delta(C(v,d))})$ is the f-vector of a simplicial complex. When $v = d + 2$, the Betti sequence of $I_{\Delta(C(v,d))}$ is $(2, 1)$, which is the f-vector of a 1-simplex.

Let $v > d + 2$ and suppose that there exists a simplicial complex $\Gamma^\sharp = \Gamma(v-1, d)$ such that $f(\Gamma^\sharp) = \beta(I_{\Delta(C(v-1,d))})$. On the other hand, since we are working on induction on d', it follows that there exists a simplicial complex $\Gamma^\flat = \Gamma(v-2, d-2)$ such that $f(\Gamma^\flat) = \beta(I_{\Delta(C(v-2,d-2))})$. We will assume that the vertex set of $\Gamma^\sharp$ and that of $\Gamma^\flat$ are disjoint.

Let x_0 be a new vertex. Again, by using the formula (1) it follows easily that

$$\beta_i(I_{\Delta(C(v,d))}) = \begin{cases} f_0(\{x_0\} * \Gamma^\sharp) + f_0(\Gamma^\flat) - 1, & i = 0, \\ f_i(\{x_0\} * \Gamma^\sharp) + f_i(\Gamma^\flat), & 1 \leq i \leq v-d-3, \\ f_{v-d-2}(\{x_0\} * \Gamma^\sharp) + f_{v-d-2}(\Gamma^\flat) - 1, & i = v-d-2, \\ 1, & i = v-d-1. \end{cases}$$

In other words,

$$\beta_i(I_{\Delta(C(v,d))}) = f_i(\{x_0\} * \Gamma^\sharp) + f_i(\Gamma^\flat) - 1, \quad i = 0, v-d-2, v-d-1.$$

Let y_0 be a vertex of $\Gamma^\flat$. Let $F \in \Gamma^\flat$ be the unique face of dimension $v-d-1$ and G a maximal proper subset of F. Then the simplicial complex

$$\Gamma(v,d) = (\{y_0\} * \Gamma^\sharp) \cup (\Gamma^\flat \setminus \{F, G\})$$

satisfies $\beta_i(I_{\Delta(C(v,d))}) = f_i(\Gamma(v,d))$ for all i, as desired. □

Conjecture 1.10. The Betti sequence arising from a pure resolution of a monomial ideal is the f-vector of a simplicial complex.

2. Edge ideals of chordal graphs

Let V be the vertex set and G a finite graph on V having no loop and no multiple edge. Let $E(G)$ denote the edge set of G. We write $S = K[\{x : x \in V\}]$ for the polynomial ring in $|V|$ variables over a field K with $\deg x = 1$ for any $x \in V$. The *edge ideal* of G is the ideal $I(G)$ of S generated by those monomials xy with $\{x,y\} \in E(G)$.

We cannot escape from the temptation to ask if the Betti sequence of the edge ideal of a finite graph can be the f-vector of a simplicial complex. Unfortunately, as was stated explicitly in Introduction, the Betti sequence of the edge ideal of the cycle of length 6 cannot be the f-vector of a simplicial complex. However, it turns out to be true that the Betti sequence of the edge ideal of a finite chordal graph can be the f-vector of a simplicial complex (Theorem 2.1). Recall that a finite graph G is *chordal* if each cycle of G of length > 3 has a chord.

Theorem 2.1. *Given an arbitrary chordal graph G, there exists a simplicial complex Δ whose f-vector $f(\Delta)$ coincides with the Betti sequence $\beta(I(G))$ of the edge ideal $I(G)$.*

The recursive-type formula ([12, Theorem 5.8]) due to Hà and Van Tuyl will be indispensable to achieve the proof of Theorem 2.1.

Let, as before, G be a finite graph on V and $E(G)$ its edge set. Given a subset $W \subset V$, the *restriction* G to W is the finite graph G_W on W whose edges are those edges $e \in E(G)$ with $e \subset W$. The *neighborhood* of a vertex v of G is the subset $N(v) \subset V$ consisting of those vertices u of G with $\{u,v\} \in E(G)$. We write $G \setminus e$, where $e \in E(G)$, for the subgraph of G which is obtained by removing e from G. The *distance* $\mathrm{dist}_G(e,e')$ of two edges $e, e' \in E(G)$ is the smallest integer $\ell \geq 0$ for which there is a sequence $e = e_0, e_1, \ldots, e_\ell = e'$, where each $e_i \in E(G)$, with $e_i \cap e_{i+1} \neq \emptyset$ for all i.

A *complete graph* on V is the finite graph on V such that $\{x,y\}$ is its edge for all $x, y \in V$ with $x \neq y$.

Lemma 2.2 (Hà and Van Tuyl)**.** *Let G be a chordal graph and $E(G)$ its edge set. Suppose that $e = \{u, v\}$ is an edge of G such that $G_{N(v)}$ is a complete graph. Let $t = |N(u) \setminus \{v\}|$ and G' the subgraph of G with*

$$E(G') = \{e' \in E(G) \,:\, \mathrm{dist}_G(e, e') \geq 3\}.$$

Then each of $G \setminus e$ and G' is chordal and

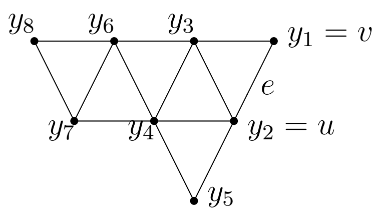
$$\beta_i(I(G)) = \beta_i(I(G \setminus e)) + \sum_{\ell=0}^{i} \binom{t}{\ell} \beta_{i-\ell-1}(I(G')) \tag{2}$$

for all $i \geq 0$, where $\beta_{-1}(I(G')) = 1$.

Remark 2.3. (a) In Dirac [9] it is proved that a finite graph G is chordal if and only if G possesses a "perfect elimination ordering." This fact guarantees the existence of a vertex v of a chordal graph G such that $G_{N(v)}$ is a complete graph.

(b) Let $N(u) = \{v, x_1, \ldots, x_t\}$. Since $G_{N(v)}$ is complete, if $\{v, z\} \in E(G)$, then $\{u, z\} \in E(G)$. In particular, if $z \notin \{u, v, x_1, \ldots, x_t\}$, then $\{v, z\} \notin E(G)$. Thus an edge e' of G satisfies $\mathrm{dist}_G(e, e') \leq 2$ if and only if $e' \cap \{u, v, x_1, \ldots, x_t\} \neq \emptyset$. Let W denote the subset of V consisting of those vertices z such that there is $e' \in E(G')$ with $z \in e'$. In particular $W \subset V \setminus \{u, v, x_1, \ldots, x_t\}$. Obviously $G' \subset G_W$. Since none of the vertices $u, v, x_1, x_2, \ldots, x_t$ belongs to W, one has $\mathrm{dist}_G(e, e') \geq 3$ for $e' \in E(G_W)$. Hence $G_W \subset G'$. Thus $G' = G_W$.

Example 2.4. Let G be the chordal graph on $\{y_1, \ldots, y_8\}$ drawn below. Let $v = y_1$, $u = y_2$, and $e = \{u, v\}$. Then $G_{N(v)}$ is a complete graph, $N(u) \setminus \{v\} = \{y_3, y_4, y_5\}$, $t = 3$ and $G' = G_{\{y_6, y_7, y_8\}}$.



The Betti sequences of $I(G)$, $I(G \setminus e)$ and $I(G')$ are

$$\beta(I(G)) = (13, 36, 47, 34, 13, 2),$$
$$\beta(I(G \setminus e)) = (12, 30, 33, 18, 4), \quad \beta(I(G')) = (3, 2).$$

We can easily check that these Betti sequences satisfy the formula (2) due to Hà and Van Tuyl. For example, since $47 = 33 + 2 \cdot \binom{3}{0} + 3 \cdot \binom{3}{1} + 1 \cdot \binom{3}{2}$, one has

$$\beta_2(I(G)) = \beta_2(I(G \setminus e)) + \binom{3}{0}\beta_1(I(G')) + \binom{3}{1}\beta_0(I(G')) + \binom{3}{2}\beta_{-1}(I(G')).$$

Lemma 2.5. *Let G be an arbitrary graph on $V = V(G)$ and W a subset of V. Then one has*

$$\beta_i(I(G)) \geq \beta_i(I(G_W))$$

for all i.

Proof. Since $I(G)$ and $I(G_W)$ are squarefree monomial ideals, there exist simplicial complexes Δ on V and Δ' on W such that $I_\Delta = I(G)$ and $I_{\Delta'} = I(G_W)$. Hochster's formula [21, Corollary 4.9, p. 64] says that

$$\beta_i(I(G)) = \beta_i(I_\Delta) = \sum_{U \subset V} \dim_K \widetilde{H}_{|U|-i-2}(\Delta_U; K),$$
$$\beta_i(I(G_W)) = \beta_i(I_{\Delta'}) = \sum_{U \subset W} \dim_K \widetilde{H}_{|U|-i-2}(\Delta'_U; K).$$

What we must prove is that $\Delta_U = \Delta'_U$ whenever $U \subset W$.

Let $F \in \Delta_U$. Then, for all $\{x, y\} \subset F$, one has $\{x, y\} \notin E(G)$. In particular $\{x, y\} \notin E(G_W)$. Thus $F \in \Delta'$ and $F \in \Delta'_U$. Conversely, let $F \in \Delta'_U$. Then, for all $\{x, y\} \subset F$, one has $\{x, y\} \notin E(G_W)$. Since $\{x, y\} \subset F \subset U \subset W$, one has $\{x, y\} \notin E(G)$. Hence $F \in \Delta$ and $F \in \Delta_U$, as desired. □

Lemma 2.6. *Let S be a polynomial ring over a field K.*

(a) *Let $I \subset S$ be a squarefree monomial ideal and x a variable of S. Then*

$$\beta_i(I) \geq \beta_i(I : x)$$

for all i.

(b) *Let I and J be monomial ideals of S and $G(I)$ (resp. $G(J)$) the minimal system of monomial generators of I (resp. J). Suppose that* $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$ *for all $u \in G(I)$ and for all $v \in G(J)$, where* $\mathrm{supp}(u)$ *is the set of variables x of S which divides u. Then, for all i, one has*

$$\beta_i(S/(I + J)) = \sum_{m=0}^{i} \beta_{i-m}(S/I)\beta_m(S/J).$$

Proof. (Sketch) (a) Let $R = S/(x-1)$ and $J = (I : x) \otimes R \subset R$. Then $\beta_i^R(J) = \beta_i^S(I : x)$, where $\beta_i^R(J)$ are the Betti numbers of J over R. Let $F_\bullet$ be a minimal graded free resolution of S/I over S. By [6, Proposition 1.1.5], it follows that $F_\bullet \otimes S/(x-1)$ is a free resolution of R/J over R. Hence $\beta_i^S(I) \geq \beta_i^R(J)$.

(b) Let $F_\bullet$ (resp. $G_\bullet$) be a minimal graded free resolution of S/I (resp. S/J). Then $F_\bullet \otimes G_\bullet$ is a minimal graded free resolution of $S/(I + J)$. □

Lemma 2.7. *Let G be an arbitrary graph on V and let W be a subset of V. Suppose that $G_{V \setminus W}$ contains edges*

$$\{u, x_1\}, \{u, x_2\}, \ldots, \{u, x_t\},$$

where $t \geq 1$ is an integer and where $u, x_1, x_2, \ldots, x_t$ are distinct vertices of G. If $\{u, z\} \notin E(G)$ for all $z \in W$, then

$$\beta_i(I(G)) \geq \sum_{m=0}^{i+1} \binom{t}{m} \beta_{i-m}(I(G_W))$$

for all $i \geq 0$, where $\beta_{-1}(I(G_W)) = 1$.

Proof. Set $V' = \{u, x_1, \ldots, x_t\}$. Lemma 2.5 together with Lemma 2.6 (a) says that

$$\beta_i(I(G)) \geq \beta_i(I(G_{V' \cup W})) \geq \beta_i(I(G_{V' \cup W}) : u).$$

Since $\{u, z\} \notin E(G)$ for all $z \in W$, it follows that

$$I(G_{V' \cup W}) : u = I(G_W) + (x_1, x_2, \ldots, x_t).$$

Then, since $V' \cap W = \emptyset$, by using Lemma 2.6 (b), one has

$$\begin{aligned} \beta_i(I(G_W) + (x_1, x_2, \ldots, x_t)) &= \beta_{i+1}(S/(I(G_W) + (x_1, x_2, \ldots, x_t))) \\ &= \sum_{m=0}^{i+1} \beta_{i+1-m}(S/I(G_W)) \beta_m(S/(x_1, x_2, \ldots, x_t)) \\ &= \sum_{m=0}^{i+1} \binom{t}{m} \beta_{i-m}(I(G_W)), \end{aligned}$$

as required. □

Let Δ be a simplicial complex on the vertex set V and let x be a new vertex. The *cone* over Δ with the vertex x is the simplicial complex

$$\text{cone}(\Delta) = \{\{x\} \cup F \ : \ F \in \Delta\} \cup \Delta$$

on $V \cup \{x\}$. Moreover, by setting $\text{cone}^0(\Delta) = \Delta$, the tth cone of Δ is defined recursively by

$$\text{cone}^t(\Delta) = \text{cone}(\text{cone}^{t-1}(\Delta)).$$

It follows that

$$f_i(\text{cone}^t(\Delta)) = \sum_{\ell=0}^{i+1} \binom{t}{\ell} f_{i-\ell}(\Delta)$$

for all i.

We are now in the position to give a proof of Theorem 2.1. Recall that the Stanley–Reisner ideal $I_\Delta \subset S$ is *squarefree lexsegment* ([1]) if, for all monomials u and v of S with $\deg u = \deg v$ and with $v <_{\text{lex}} u$ such that $v \in I_\Delta$, one has $u \in I_\Delta$, where $<_{\text{lex}}$ is the lexicographic order induced by a (fixed) ordering of the variables of S. Given a simplicial complex Δ, there is a unique simplicial complex Δ^{lex} such that $I_{\Delta^{\text{lex}}}$ is squarefree lexsegment with $f(\Delta) = f(\Delta^{\text{lex}})$.

Proof of Theorem 2.1. Our proof will proceed by using induction on the number of edges of G. If G possesses only one edge $\{x, y\}$, then $I(G) = (xy)$ and

$$\beta_i(I(G)) = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0. \end{cases}$$

Thus its Betti sequence is equal to the f-vector of a 0-simplex.

Now, suppose that G possesses at least two edges and that, for an arbitrary chordal graph Γ with $|E(\Gamma)| < |E(G)|$, the Betti sequence $\beta(I(\Gamma))$ is the f-vector $f(\Delta_\Gamma)$ of a simplicial complex Δ_Γ.

Let $e = \{u, v\}$ be an edge of G such that $G_{N(v)}$ is complete. Work with the same notation as in Lemma 2.2 and in Remark 2.3 (b). Note that $W = \{z \in V : z \in e' \text{ for some } e' \in E(G')\}$ and $G' = G_W$. Then one has

$$\beta_i(I(G)) = \beta_i(I(G \setminus e)) + \sum_{\ell=0}^{i} \binom{t}{\ell} \beta_{i-\ell-1}(I(G_W)).$$

Since each of $G \setminus e$ and G_W is a subgraph of G with $e \notin E(G \setminus e)$ and $e \notin E(G_W)$, the hypothesis of induction guarantees the existence of simplicial complexes $\Delta_{G\setminus e}$ and Δ_{G_W} such that

$$f_i(\Delta_{G\setminus e}) = \beta_i(I(G \setminus e)), \quad f_i(\Delta_{G_W}) = \beta_i(I(G_W)).$$

Thus what we must prove is the existence of a simplicial complex Δ with

$$f_i(\Delta) = f_i(\Delta_{G\setminus e}) + \sum_{\ell=0}^{i} \binom{t}{\ell} f_{i-\ell-1}(\Delta_{G_W}). \tag{3}$$

It follows from Lemma 2.7 that

$$\beta_i(I(G \setminus e)) \geq \sum_{m=0}^{i+1} \binom{t}{m} \beta_{i-m}(I(G_W)).$$

In other words,

$$f_i(\Delta_{G\setminus e}) \geq \sum_{m=0}^{i+1} \binom{t}{m} f_{i-m}(\Delta_{G_W}) = f_i(\mathrm{cone}^t(\Delta_{G_W})).$$

Thus, by choosing $\Delta_{G\setminus e}$ for which $I_{\Delta_{G\setminus e}}$ is squarefree lexsegment, we assume that $\Delta_{G\setminus e}$ contains a subcomplex Δ' whose f-vector coincides with that of $\mathrm{cone}^t(\Delta_{G_W})$.

We introduce the simplicial complex Δ by setting

$$\Delta = \Delta_{G\setminus e} \cup \mathrm{cone}(\Delta'),$$

where the new vertex of $\text{cone}(\Delta')$ cannot be a vertex of $\Delta_{G\setminus e}$. Then

$$\begin{aligned} f_i(\Delta) - f_i(\Delta_{G\setminus e}) &= f_i(\text{cone}(\Delta')) - f_i(\Delta') \\ &= f_{i-1}(\Delta') \\ &= f_{i-1}(\text{cone}^t(\Delta_{G_W})) \\ &= \sum_{\ell=0}^{i} \binom{t}{\ell} f_{i-\ell-1}(\Delta_{G_W}). \end{aligned}$$

Thus the simplicial complex satisfies the equality (3), as desired. □

3. Gorenstein Monomial Ideals

We now turn to the discussion on Betti sequences of Gorenstein monomial ideals. Let, as before, $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K with $\deg x_i = 1$ for any i. Recall that a homogeneous ideal $I \subset S$ is Gorenstein if S/I is a Gorenstein ring. If $I \subset S$ is Gorenstein, then its Betti sequence $\beta(I) = (\beta_0(I), \beta_1(I), \ldots, \beta_p(I))$ is symmetric, that is, $\beta_i(I) = \beta_{p-1-i}(I)$ for all i, where $p = \text{proj}\dim(I)$ and where $\beta_{-1}(I) = 1$.

Let $I \subset S$ be a Gorenstein monomial ideal with $\text{proj}\dim(I) = p$. If $p = 1$, then $\beta(I) = (2, 1)$ by the Hilbert–Burch theorem [6, Theorem 1.4.17]. If $p = 2$, then there exists an odd integer $m \geq 3$ such that $\beta(I) = (m, m, 1)$ by the structure theorem due to Buchsbaum and Eisenbud ([6, Theorem 3.4.1]). In fact, these facts characterize the Betti numbers of Gorenstein (monomial) ideals with $\text{proj}\dim(I) \leq 2$. For example, to prove the sufficiency, let I be the Stanley–Reisner ideal of the boundary complex of the cyclic $2m$-polytope with $2m + 3$ vertices. Then I is a Gorenstein ideal with $\beta(I) = (2m + 3, 2m + 3, 1)$ for all $m \geq 1$ by the formula (1).

Let $p = 3$. Let $I \subset S$ be a Gorenstein monomial ideal with $\text{proj}\dim(I) = 3$. Since $(\beta_{-1}(I), \beta_0(I), \beta_1(I), \beta_2(I), \beta_3(I))$, where $\beta_{-1}(I) = 1$, is symmetric and since $\sum_{i=-1}^{3}(-1)^i\beta_i(I) = 0$, it follows that there exists an integer m such that $\beta(I) = (m + 1, 2m, m + 1, 1)$. Since I is a monomial ideal, the Taylor resolution of I says that $m = \beta_0(I) - 1 \geq \text{proj}\dim(I) = 3$. Since $(m, m, 1)$ is the f-vector of a simplicial complex for $m \geq 3$, it follows from Lemma 1.1 that $\beta(I)$ is the f-vector of an acyclic simplicial complex.

Example 3.1. Let $I = (x_1x_4, x_1x_5, x_2x_6, x_3x_7, x_4x_6, x_4x_7, x_2x_3x_5)$. Then I is Gorenstein and $\beta(I) = (7, 12, 7, 1) = (6 + 1, 2 \times 6, 6 + 1, 1)$.

More precisely, we can characterize the Betti numbers of Gorenstein monomial ideals I with $\text{proj}\dim(I) = 3$. Recall that a monomial ideal $I \subset S$ is *strongly stable* if, for all monomials $u \in I$ and for all $j < i$ such that x_i divides u, one has $ux_j/x_i \in I$.

Theorem 3.2. *Let $\beta = (m+1, 2m, m+1, 1)$, where m is an integer with $m \geq 3$. Then there exists a Gorenstein monomial ideal I of a polynomial ring with $\beta(I) = \beta$ if and only if $m \neq 4$.*

Proof. **("If")** Let $m \geq 3$ be odd. Then there exists a Gorenstein monomial ideal $J \subset S$ with $\beta(J) = (m, m, 1)$. Let y be a new variable and $S' = S[y]$. Then the ideal $I = J + (y)$ is a Gorenstein monomial ideal with $\operatorname{proj\,dim}(I) = 3$ and $\beta_0(I) = m+1$.

Let $m \geq 6$ be even. Example 3.1 yields an example of $m = 6$. Now, let $m = 2k + 6 \geq 8$ be even. Given a strongly stable ideal $J \subset R = K[x_1, \ldots, x_p]$ such that R/J is of finite length, it follows from [18, Theorem 9.6] and [19, Theorem 5.3] that there exists a Gorenstein squarefree monomial ideal $I_{(J)}$ for which $\beta_i(S/I_{(J)}) = \beta_i(R/J) + \beta_{p+1-i}(R/J)$ for all i. Let J be the strongly stable ideal

$$J = (x_1^2, x_1x_2, x_1x_3, x_2^{k+1}, x_2^k x_3, \cdots, x_3^{k+1}) \subset K[x_1, x_2, x_3].$$

Eliahou–Kervaire formula says that $\beta_0(I_{(J)}) = \beta_0(J) + \beta_2(J) = 2k + 7$, as required.

("Only If") We show, in general, that if $I \subset S$ is a Gorenstein monomial ideal with $\operatorname{proj\,dim}(I) = p - 1 \geq 3$, then $\beta_0(I) \neq p + 1$. Let $G(I)$ be the set of minimal monomial generators of I. Suppose, on the contrary, that there exists a Gorenstein monomial ideal with $\operatorname{proj\,dim}(I) = p - 1 \geq 3$ and $\beta_0(I) = p + 1$. By taking the polarization ([6, Lemma 4.2.16]) of I, we assume that I is squarefree. Let x_i be a variable such that $x_i \notin G(I)$. Let Δ (resp. Δ') denote the simplicial complex whose Stanley–Reisner ideal is I (resp. $I : x_i$). Note that Δ may not contain n vertices since I may contain a variable. Then Δ' is the star ([6, Definition 5.3.4]) of Δ of the face $\{i\}$. Hence $I : x_i$ is a Gorenstein ideal with $\dim(S/I) = \dim(S/(I : x_i))$. In particular $\operatorname{proj\,dim}(I) = \operatorname{proj\,dim}(I : x_i)$. Thus, in case of $\beta_0(I) = \beta_0(I : x_i)$, we replace I with $I : x_i$. Hence, for each variable $x_k \notin G(I)$, we assume that $\beta_0(I : x_k) < p + 1$. In particular, for each $x_k \notin G(I)$, since $\operatorname{proj\,dim}(I) = \operatorname{proj\,dim}(I : x_k) \leq \beta_0(I : x_k) - 1$, it follows that $\beta_0(I : x_k) = p$ and $I : x_k$ is a complete intersection.

Let $G(I) = \{u_1, \ldots, u_{p+1}\}$. Since I is not a complete intersection, $G(I)$ is not a set of variables. Suppose that $x_1 \notin G(I)$ and x_1 divides u_1. Since $\beta_0(I : x_1) = p$, there exists u_k with $k \neq 1$ such that u_1/x_1 divides u_k. We may assume $k = p + 1$. Let $u_1 = x_1 x_F$ and $u_{p+1} = x_F x_G$, where $x_F = \prod_{i \in F} x_i$ with $F \subset [n]$. Then

$$I : x_1 = (\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_p),$$

where $\tilde{u}_k = u_k/x_1$ (resp. $\tilde{u}_k = u_k$) if x_1 divides (resp. does not divide) u_k. In particular $\tilde{u}_1 = x_F$. Since $I : x_1$ is a complete intersection, it follows that

$$\operatorname{supp}(\tilde{u}_s) \cap \operatorname{supp}(\tilde{u}_t) = \emptyset \tag{4}$$

if $s \neq t$, where $\operatorname{supp}(\tilde{u}_s)$ stands for the set of variables x_k which divides $\tilde{u}_s$. If there is $2 \leq k \leq p$ with $u_k = \tilde{u}_k$, then, since $\beta_0(I : x_j) = p$ for all $x_j \in \operatorname{supp}(u_k)$, it follows from (4) that u_k must divide u_{p+1}, a contradiction. Thus $\tilde{u}_k = u_k/x_1$ for each $1 \le k \le p$. Let $j \in F$. Then, by (4), $x_j \notin \mathrm{supp}(u_k)$ for $k = 2, \ldots, p$. Since $\beta_0(I : x_j) = p$, there is k with $2 \le k \le p$ such that either u_1/x_j or u_{p+1}/x_j must divide u_k. If u_1/x_j divides u_k, then $u_1 = x_1 x_j$ by (4). Thus $\beta_0(I : x_j) = p = 2$, a contradiction. If $u_{p+1}/x_j = x_G x_F / x_j$ divides u_k, then, again by (4), one has $u_1 = x_1 x_F = x_1 x_j$ and $p = 2$, a contradiction. □

The technique appearing in the "If" part of the proof of Theorem 3.2 together with the result shown in the "Only If" part of Theorem 3.2 yields the following

Corollary 3.3. *Fix integers $m \ge 4$ and $p \ge 3$. Then there exists a Gorenstein monomial ideal I of $K[x_1, \ldots, x_n]$, where n is enough large, with $\beta_0(I) = m$ and* $\mathrm{proj\,dim}(I) = p$ *if and only if $m \ge p + 1$ with $m \ne p + 2$.*

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