HILBERT SCHEMES AND BETTI NUMBERS OVER A CLEMENTS-LINDSTRÖM RING

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Abstract: We show that the Hilbert scheme, that parametrizes all ideals with the same Hilbert function over a Clements-Lindtröm ring W, is connected. More precisely, we prove that every graded ideal is connected by a sequence of deformations to the lex-plus-powers ideal with the same Hilbert function. Our result is an analogue of Hartshorne's theorem that Grothendieck's Hilbert scheme is connected; however our proof is entirely different, since Hartshorne's deformations (distractions) do not work over W. We also prove a conjecture by Gasharov, Hibi, and Peeva that the lex ideal attains maximal Betti numbers among all graded ideals in W with a fixed Hilbert function.

1. Introduction

Throughout the paper $S = k[x_1, \ldots, x_n]$ is a polynomial ring graded by $\deg(x_i) = 1$ for all i, over an algebraically closed field k of characteristic zero. One of the central results in commutative algebra is Macaulay's Theorem [Ma], which characterizes the possible Hilbert functions of graded ideals in S. The key idea is that for every graded ideal in S there exists a lex ideal with the same Hilbert function. Lex ideals also play important role in the study of Hilbert schemes.

Grothendieck's Hilbert scheme, introduced by Grothendieck [Gr], parametrizes subschemes of $\mathbf{P}^r$ with a fixed Hilbert polynomial. The structure of Grothendieck's Hilbert scheme is known to be quite complicated. The following result of Hartshorne is the main known positive structural result.

Theorem 1.1. [Ha] *The Hilbert scheme, that parametrizes subschemes of $\mathbf{P}^r$ with a fixed Hilbert polynomial, is connected.*

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A minor modification of its proof shows that:

Theorem 1.2. *The Hilbert scheme $\mathcal{H}_S(h)$, that parametrizes all graded ideals in S with a fixed Hilbert function h, is linearly connected. Every graded ideal in the polynomial ring S is connected by a sequence of deformations to the lex ideal with the same Hilbert function.*

Theorem 1.2 implies Theorem 1.1 since if L and L' are two lexicographic ideals with the same Hilbert polynomial then L and L' represent the same point on the Hilbert scheme (because $L_d = L'_d$ for $d \gg 0$).

It is known that Theorems 1.1 and 1.2 hold over an exterior algebra. Results on the structure of the Hilbert scheme over an exterior algebra are proved in [AS] and [PS2].

The structure of the Hilbert scheme over S is known to be very complicated, and it seems that this discouraged people from trying to study Hilbert schemes over quotient rings as the proofs and constructions become significantly more intricate. The Clements-Lindström rings are a natural class of quotient rings to consider because Macaulay's Theorem holds over them. This paper is focused on such rings.

Let $P = (x_1^{a_1}, \cdots, x_n^{a_n})$, with $a_1 \le a_2 \le \cdots \le a_n \le \infty$ (here $x_i^\infty = 0$) and set $W = S/P$. We say that W is a *Clements-Lindström ring*. We refer to $x_1^{a_1}, \cdots, x_n^{a_n}$ as the *P-powers*. The generalization of the notion of a lex ideal to W is the notion of a lex+P ideal, defined in Section 2. The well known Clements-Lindström Theorem [CL] characterizes the possible Hilbert functions of graded ideals in the quotient ring W; Macaulay's Theorem [Ma] covers the particular case when $W = S$. The Clements-Lindström Theorem states that for every graded ideal in S containing P there exists a lex+P ideal in S with the same Hilbert function; equivalently, for every graded ideal in the Clements-Lindström ring W there exists a lex ideal with the same Hilbert function. We consider the Hilbert scheme $\mathcal{H}_W(h)$, that parametrizes all graded ideals in W with a fixed Hilbert function h. Equivalently, this Hilbert scheme parametrizes all graded ideals in S containing the P-powers and with a fixed Hilbert function. In Section 3, we define that a deformation is a P-deformation if it connects ideals containing the P-powers.

In Section 3, we prove:

Theorem 1.3. *The Hilbert scheme $\mathcal{H}_W(h)$, that parametrizes all graded ideals in W with a fixed Hilbert function h, is linearly connected. Every graded ideal in the polynomial ring S that contains the P-powers, is connected by a sequence of P-deformations to the lex+P ideal with the same Hilbert function.*

Note that the Hilbert scheme $\mathcal{H}_W(h)$ is much smaller than Grothendieck's Hilbert scheme that parametrizes all graded ideals in S with a fixed Hilbert function (this is a variation of the classical Grothendieck's Hilbert scheme). Also note that generic changes of coordinates, used by Hartshorne, do not work over W since they destroy the P-powers. In this situation, it is quite surprising that $\mathcal{H}_W(h)$ is connected.

We prove Theorem 1.3 as follows. Given an ideal V on the Hilbert scheme $\mathcal{H}_W(h)$, we construct an explicit path from V to the lex+P ideal on $\mathcal{H}_W(h)$. As we mentioned above, Hartshorne's proof [Ha] relies on generic change of coordiantes. Unfortunately, generic change of coordiantes does not work in W because it destroys the P-powers. We overcome this difficulty by constructing walks on $\mathcal{H}_W(h)$ in an entirely different way than Hartshorne's. Our walks consist of repeatedly performing the following two steps. (1) We construct a walk based on the idea of "filling gaps". This idea was used by Peeva and Stillman in [PS] over an exterior algebra; the construction over W is more intricate because the gaps have a more complicated form than those over an exterior algebra. (2) We construct walks using special changes of coordinates.

As a corollary of Theorem 1.2, Macaulay's Theorem was generalized to Betti numbers by Bigatti, Hulett, and Pardue as follows:

Theorem 1.4. *Every lex ideal in S attains maximal Betti numbers among all graded ideals with the same Hilbert function.*

The above result holds over an exterior algebra as well. Aramova, Herzog, and Hibi [AHH] prove that every lex ideal in an exterior algebra attains maximal Betti numbers among all graded ideals with the same Hilbert function. It was conjectured by Gasharov, Hibi, and Peeva [GHP] that Theorem 1.4 holds over the Clements-Lindström ring W. In Section 4, we prove the conjecture:

Theorem 1.5. *Every lex ideal in W attains maximal Betti numbers among all graded ideals with the same Hilbert function.*

Note that Theorem 1.4 is about finite resolutions, while Theorem 1.5 is about infinite ones.

The walks on the Hilbert scheme that we construct in Section 3, do not give information on how the Betti numbers change along the walk. In order to prove Theorem 1.5 we construct special changes of coordinates and use them to build a construction that starting with a monomial ideal yields a lex-closer ideal with bigger Betti numbers. The construction may not yield a path on the Hilbert scheme between the two ideals.

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2. Preliminaries

Here we recall and introduce several definitions and notation, which will be used in the next sections.

Throughout this section all ideals and monomials live in the polynomial ring S.

We say that a monomial $m \in S$ is *P-free* if its image in the quotient ring W is non-zero, that is, for each $1 \leq i \leq n$ we have that $x_i^{a_i}$ does not divide m. If P is generated by the squares of the variables, then the P-free monomials are called squarefree.

We say that a graded ideal I is an *ideal+P* if it contains the ideal P. Furthermore, a monomial+P ideal is a monomial ideal containing P. Such an ideal M has a unique minimal system of monomial generators that consists of P-free monomials and some of the P-powers. We denote this system of generators by $\mathrm{mingens}(M)$.

We order the variables $x_1 > \ldots > x_n$. Order the monomials in each degree lexicographically. Denote by $\succ_{lex}$ the homogeneous lexicographic order on the monomials; for simplicity, we call it lex-order. For a monomial $m \neq 1$ set

$$\max(m) = \max\{i \in \mathbf{N} \mid x_i / m\} \quad \text{and} \quad \min(m) = \min\{i \in \mathbf{N} \mid x_i / m\},$$

where x_i / m means that the variable x_i divides the monomial m.

An ideal M is called *Borel+P* if it is generated by monomials, contains P, and the following property is satisfied: if m is a P-free monomial in M, a variable x_i divides m, and $1 \leq j \leq i$, then $\dfrac{x_j m}{x_i} \in M$.

A k-vector space E spanned by monomials of the same degree j is called *lex-segment+P* if it contains P_j and the following property is satisfied: if $m \in E$ is a P-free monomial and $c \succ_{lex} m$ is a monomial of $\deg(c) = j$, then $c \in E$.

An ideal L is called *lex+P* if for each $j \geq 0$ the vector space L_j is spanned by a lex-segment+P. Clearly, $L \supseteq P$. Note that the ideal P is lex+P.

Definition 2.1. *A P-free monomial m is called a gap in a monomial+P ideal M if $m \notin M$ and M contains a lex-smaller P-free monomial of the same degree.*

Definition 2.2. *Let M and M' be two monomial+P ideals with the same Hilbert function. We say that M is lex-closer than M' is, if there exists a degree r such that the following conditions are satisfied:*
(1) $M_j = M'_j$ for each $j < r$
(2) Let $g_1, \ldots, g_p$ and $g'_1, \ldots, g'_{p'}$ be the gaps of M_r and M'_r, respectively, ordered lexicographically in decreasing order. Then $g'_j \succ_{lex} g_j$ for the first j for which the j'th gaps are different.

3. Connectedness of the Hilbert scheme

3.1. Reduction to the Borel+P case.

Definition 3.1.1. Consider the ring $\tilde{S} = S \otimes k[t]$. Let $\tilde{C}$ be an ideal in $\tilde{S}$ such that $\tilde{S}/\tilde{C}$ is flat as a $k[t]$-module. For $\alpha \in k$, the quotient $\tilde{S}/\tilde{C} \otimes (k[t]/(t-\alpha))$ is denoted $(\tilde{S}/\tilde{C})_\alpha$ and is called the *fiber over* α. For any $\alpha, \beta \in k$ we say that the fibers $(\tilde{S}/\tilde{C})_\alpha$ and $(\tilde{S}/\tilde{C})_\beta$ are *connected by a deformation over* $\mathbf{A}^1_C$. We say that two ideals C and C' in S are *connected by a sequence of deformations over* $\mathbf{A}^1_k$ if S/C and S/C' are connected by a sequence of deformations over $\mathbf{A}^1_k$. For simplicity, we often say "deformation" instead of "deformation over $\mathbf{A}^1_k$". We have a P*-deformation* if $\tilde{C} \supseteq P$; in this case, $\tilde{C}_\alpha \supseteq P$ for every α.

It is well known, cf. [Ei, Chapter 15], that:

Lemma 3.1.2. *Let U be a graded ideal in S that contains the powers P. Fix a monomial order $\prec$. The initial ideal $\text{in}_\prec U$ and U are connected by a P-deformation.*

Construction 3.1.3. Fix a $1 \leq j \leq n$.

The j'th *polarization* of a monomial $m = \prod x_i^{e_i}$ is $\text{pol}_j(m) = m$ if j does not divide m, and otherwise

$$\text{pol}_j(m) = \left(\prod_{i \neq j} x_i^{e_i} \right) (x_j y_1 \cdots y_{e_j - 1}),$$

where the variables y_i are new variables. Let M be a monomial ideal in S. Let s be the largest power of x_j occuring in a minimal monomial generator of M. Set $\bar{S} = k[x_1, \cdots, x_n, y_1, \cdots, y_{s-1}]$. Then for every monomial $m \in \text{mingens}(M)$, we have

$\mathrm{pol}_j(m) \in \bar{S}$. The j'th *polarization* of M is the ideal M_{pol} of $\bar{S}$ generated by the j'th polarizations of the minimal monomial generators of M, that is,

$$M_{pol} = \big(\mathrm{pol}_j(m) \,|\, m \in \mathrm{mingens}(M)\big)\,.$$

Assume that $a_j < \infty$. Let ζ be a primitive a_j'th root of unity (e.g., $\zeta = \cos\frac{2\pi}{a_j} + \sqrt{-1}\sin\frac{2\pi}{a_j}$). Let ϕ_{lj} be the autormorphism of $\bar{S}$ given by $\phi_{lj}(x_i) = x_i$ for $i \neq j$, $\phi_{lj}(x_j) = x_l - x_j$, and $\phi_{lj}(y_i) = x_l - \zeta^i x_j + y_i$ for all y_i. Set

$$M'' = \Bigg(f(x_1, \ldots, x_n, 0, \ldots, 0) \,|\, f = \phi(\mathrm{pol}_j(m)) \text{ for } \; m \in \mathrm{mingens}(M) \Bigg) \subset \bar{S}\,,$$

and denote by M' the ideal in S generated by the same generators.

Lemma 3.1.4. *We use the notation in Construction 3.1.3. Let rlex be a revlex monomial order in $\bar{S}$ so that the y-variables are smaller than the x-variables. Suppose that M is a monomial ideal that satisfies the following conditions:*
(1) The minimal monomial generators of the ideal $\mathrm{in}_{rlex}\phi(M_{pol})$ are monomials in S.
(2) The ideals M and $\big(\mathrm{in}_{rlex}\phi(M_{pol})\big) \cap S$ have the same Hilbert function.
(3) Both M and M'' contain the P-powers.
Then M and $\big(\mathrm{in}_{rlex}\phi(M_{pol})\big) \cap S$ are connected by a sequence of two P-deformations.

Proof: Note that the ideal M' contains the P-powers by (3).

By (1), it follows that M'' and $\phi(M_{pol})$ have the same initial ideal with respect to $rlex$. Hence, $\mathrm{in}_{rlex}M' = \big(\mathrm{in}_{rlex}\,\phi(M_{pol})\big) \cap S$. By Lemma 3.1.2, it follows that the ideals M' and $\mathrm{in}_{rlex}M' = \big(\mathrm{in}_{rlex}\,\phi(M_{pol})\big) \cap S$ are connected by a P-deformation.

On the other hand, consider a lex order **lex** on S such that $x_j >_{\mathbf{lex}} x_l$. By construction, it follows that $M \subseteq \mathrm{in}_{\mathbf{lex}}M'$. By (2) we conclude that M' and M have the same Hilbert function. Hence, $M = \mathrm{in}_{\mathbf{lex}}M'$. Therefore, M and M' are connected by a P-deformation by Lemma 3.1.2. □

Applying Lemma 3.1.4 to the results by Mermin and Murai in [MM, Section 3] we obtain the following result.

Proposition 3.1.5. *Let J be a monomial+P ideal which is not Borel+P. There exists a Borel+P ideal B which is lex-closer than J and which is connected to J by a sequence of P-deformations; in particular, B has the same Hilbert function as J.*

Remark. Mermin and Murai did not state that B is lex-closer than U. However, it follows from their proof that the construction given in [MM, Section 3] always gives a monomial+P ideal which is lex-closer than the original ideal.

3.2. Filling gaps in a Borel+P ideal. In the rest of Section 3, all ideals and monomials live in the polynomial ring S, and B stands for a Borel+P ideal that is not lex+P.

Construction 3.2.1. If $m = x_1^{e_1} \dots x_n^{e_n}$ is a monomial and $1 \le j \le \deg(m)$ is an integer, we define the *j-th beginning* of m to be the monomial $\mathrm{begin}_j(m) = x_1^{e_1} \dots x_{i-1}^{e_{i-1}} x_i^{\mu}$, where $1 \le \mu \le e_i$ and $deg(x_1^{e_1} \dots x_{i-1}^{e_{i-1}} x_i^{\mu}) = j$.

Set

$$q = \min\{h \in \mathbf{N} \,|\, B_h \text{ is not spanned by a lex-segment+}P\}\,.$$

Furthermore, denote by $\tilde{g}$ the lex-greatest gap in B_q, and denote by $\tilde{b}$ the lex-greatest P-free monomial in B_q that is lex-smaller than $\tilde{g}$. We can write $\tilde{g} = dg'$ and $\tilde{b} = db'$, where d, g', b' are P-free monomials and either $d = 1$, or $\max(d) \le \min(g')$ and $\max(d) < \min(b')$.

Choose the minimal number $l \in \mathbf{N}$ so that the set of monomials

$$\begin{aligned} \mathcal{C}_j = \big\{\, \mathrm{begin}_j(\tilde{b})u \in \mathrm{mingens}(B) \,|\, & \mathrm{begin}_j(\tilde{b})u \text{ is a } P\text{-free monomial, } j > \deg(d), \\ & \mathrm{begin}_j(\tilde{g})u \notin B, \\ & \min(u) \ge \max(\mathrm{begin}_j(\tilde{g})) \text{ if } u \ne 1 \,\big\} \end{aligned}$$

is not empty. Set $\mathcal{C} = \mathcal{C}_l$, and $b = \mathrm{begin}_l(\tilde{b})$ and $g = \mathrm{begin}_l(\tilde{g})$. We form the binomial ideals

$$\begin{aligned} T &= \Big(\{\, bu - gu \,|\, bu \in \mathcal{C}\},\ \mathrm{mingens}(B) \setminus \{bu \,|\, bu \in \mathcal{C}\}\Big) \\ N &= T + (x_h^{a_h} \,|\, x_h^{a_h} \notin T,\, 1 \le h \le n\,)\,, \end{aligned}$$

where b and g are fixed, and u varies. Set $J = \mathrm{in}_{\prec_{lex}}(N)$.

The main result in this section is the following proposition.

Proposition 3.2.2. *Let B be a Borel+P ideal which is not lex+P. The monomial+P ideal J, constructed in 3.2.1, is lex-closer than B and is connected to B by a sequence of P-deformations. In particular, the ideal J has the same Hilbert function as B.*

The proposition is proved in a series of lemmas and constructions.

Notation introduced in a construction or in the statement of a lemma, will be used throughout the rest of the section.

3.3. The first gap in B.

Lemma 3.3.1. *The P-free monomials $\tilde{g}$ and $\tilde{b}$, defined in Construction 3.2.1, have the form*

$$\tilde{g} = d\, x_i^{\alpha_1}\, x_{i_2}^{\alpha_2}\, \ldots x_{i_s}^{\alpha_s}$$
$$\tilde{b} = d\, x_{i+1}^{a_{i+1}-1}\, x_{i+2}^{a_{i+2}-1}\, \ldots x_{i+p-1}^{a_{i+p-1}-1}\, x_{i+p}^{\beta}\,,$$

where the following conditions are satisfied:

- d *is a P-free monomial, and* $\max(d) \le i$ *if* $d \neq 1$,
- $1 \le i < i_2 < \ldots < i_s \le n$,
- $p \ge 1$,
- $\beta, \alpha_1, \ldots, \alpha_s \in \mathbf{N} \setminus 0$,
- $\beta + \sum_{1 \le h \le p-1} (a_h - 1) = \sum_{1 \le h \le s} \alpha_h$,
- $\beta \le a_{i+p} - 1$.

Proof: We can write the monomials $\tilde{g}$ and $\tilde{b}$ in the form

$$\tilde{g} = d\, x_i^{\alpha_1}\, x_{i_2}^{\alpha_2}\, \ldots x_{i_s}^{\alpha_s}$$
$$\tilde{b} = d\, x_{j_1}^{\beta_1}\, x_{j_2}^{\beta_2}\, \ldots x_{j_r}^{\beta_r}$$

where $\beta_1, \ldots, \beta_r, \alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbf{N} \setminus 0$, $i \neq j_1$, $1 \le i < i_2 < \ldots < i_s \le n$, $1 \le j_1 < j_2 < \ldots < j_r \le n$, and d is a monomial such that either $d = 1$, or $\max(d) \le i$ and $\max(d) \le j_1$.

Since $\tilde{g} \succ_{lex} \tilde{b}$, it follows that $i < j_1$. Hence each of the numbers $j_1, j_2, \ldots, j_s$ is greater or equal to $i+1$. Choose p to be the biggest integer for which the difference $\sum_{1 \le t \le s} \alpha_h - \sum_{1 \le f \le p-1} (a_{i+f} - 1)$ is positive. Set

$$\beta = \sum_{1 \le h \le s} \alpha_h - \sum_{1 \le f \le p-1} (a_{i+f} - 1)\,.$$

As $\tilde{b} \in B$ is a P-free monomial and the ideal B is Borel+P, it follows that the monomial

$$m := d\, x_{i+1}^{a_{i+1}-1}\, x_{i+2}^{a_{i+2}-1}\, \ldots x_{i+p-1}^{a_{i+p-1}-1}\, x_{i+p}^{\beta} \in B\,.$$

Note that this monomial is P-free since $\max(d) \leq i < i+1$. Since m is the lex-greatest P-free monomial in B_q that is lex-smaller than $\tilde{g}$ and since $i \neq j_1$, we conclude that

$$\tilde{b} = d\, x_{i+1}^{a_{i+1}-1}\, x_{i+2}^{a_{i+2}-1}\, \dots x_{i+p-1}^{a_{i+p-1}-1} x_{i+p}^{\beta}$$

as desired. $\square$

Lemma 3.3.2. *We have that either* $\max(\tilde{b}) < \max(\tilde{g})$, *or* $\max(\tilde{b}) = \max(\tilde{g})$ *and* $\beta < \alpha_s$. *In the latter case* $\max(\frac{\tilde{b}}{x_{i+p}^{\beta}}) < \max(\frac{\tilde{g}}{x_{i+p}^{\beta}})$.

Proof: We have that $\max(\tilde{g}) = i_s$ and $\max(\tilde{b}) = i+p$ by Lemma 3.3.1. Suppose that the inequality $i_s \leq i+p$ holds. Then $i_{s-r} \leq i+p-r$ for every $0 \leq r \leq p-1$.

Suppose that either $i_s < i+p$, or $i_s = i+p$ and $\beta \geq \alpha_s$. Since $\tilde{b} \in B$ and B is Borel+P, it follows that $\tilde{g} \in B$ because $a_1 \leq \ldots \leq a_n$. This is a contradiction since $\tilde{g}$ is a gap by assumption. $\square$

Lemma 3.3.3. *We have that* $\tilde{b} \in$ *mingens*(B).

Proof: Suppose that $\tilde{b}$ is not a minimal monomial generator of B. Therefore, $\tilde{b} \in S_1 B_{q-1}$. As B_{q-1} is spanned by a lex-segment+P_{q-1}, it follows that $S_1 B_{q-1}$ is spanned by a lex-segment+P_q. As both $\tilde{g}$ and $\tilde{b}$ are P-free monomials and $\tilde{g} \succ_{lex} \tilde{b}$, we conclude that $\tilde{g} \in S_1 B_{q-1}$. This is a contradiction because $\tilde{g}$ is a gap by assumption. $\square$

Lemma 3.3.4. *We have that* $\tilde{g} x_h \in B$ *for every number* $h < \max(\tilde{g})$.

Proof: If the monomial $\tilde{g} x_h$ is not P-free, then we are done. Suppose that it is P-free. Let $h < i_s$ be a natural number. We have that $\dfrac{\tilde{g}}{x_{\max(\tilde{g})}} x_h \succ_{lex} \tilde{g}$. Hence $\dfrac{\tilde{g}}{x_{\max(\tilde{g})}} x_h \in B$ because $\tilde{g}$ is the lex-greatest (first) gap in B_q. Therefore, $\tilde{g} x_h \in B$. $\square$

3.4. A binomial plus P ideal.

Consider $\tilde{g}$ and $\tilde{b}$ introduced in Construction 3.2.1. Recall that

$$\begin{aligned} \mathcal{C}_j = \Big\{ \operatorname{begin}_j(\tilde{b})u \in \operatorname{mingens}(B) \,|\, & \operatorname{begin}_j(\tilde{b})u \text{ is a } P\text{-free monomial, } j > \deg(d), \\ & \operatorname{begin}_j(\tilde{g})u \notin B, \\ & \min(u) \geq \max(\operatorname{begin}_j(\tilde{g})) \text{ if } u \neq 1 \Big\}. \end{aligned}$$

Lemma 3.4.1. *There exists a* $j \in \mathbf{N}$ *such that* $\mathcal{C}_j \neq \emptyset$.

Proof: By Lemma 3.3.3 we have that $\tilde{b} \in \mathcal{C}$ with $u = 1$ and $\operatorname{begin}_j(\tilde{b}) = \tilde{b}$. $\square$

Lemma 3.4.2. *If* $bu \in \mathcal{C}$*, then either* $\min(u) \geq \max(g)$ *or* $u = 1$.

Proof: If $u \neq 1$, then we have that $\min(u) \geq \max(\mathrm{begin}_l(\tilde{g})) = \max(g)$ by Construction 3.2.1.

□

Construction 3.4.3. Choose the minimal number $l \in \mathbf{N}$ so that $\mathcal{C}_l \neq \emptyset$. Recall by 3.2.1. that $\mathcal{C} = \mathcal{C}_l$, and $b = \mathrm{begin}_l(\tilde{b})$ and $g = \mathrm{begin}_l(\tilde{g})$. By Lemma 3.3.1 it follows that the l-th beginning of $\tilde{b}$ has the form

$$b = \mathrm{begin}_l(\tilde{b}) = d\, x_{i+1}^{a_{i+1}-1}\, x_{i+2}^{a_{i+2}-1}\, \ldots x_{i+t-1}^{a_{i+t-1}-1} x_{i+t}^{\gamma}$$

for some $1 \leq t \leq p$ and such that either $t = p$ and $1 \leq \gamma \leq \beta$, or $t \neq p$ and $1 \leq \gamma \leq a_{i+t} - 1$. Furthermore, the l-th beginning of $\tilde{g}$ has the form

$$g = \mathrm{begin}_l(\tilde{g}) = d\, x_i^{\alpha_1}\, x_{i_2}^{\alpha_2}\, \ldots x_{i_r}^{\nu}\,,$$

where $1 \leq \nu \leq \alpha_r$. The integers t and r above are defined by the condition that the beginning monomial should have degree l.

Lemma 3.4.4. *If* $\dfrac{b}{x_{\max(b)}} v \in \mathrm{mingens}(B)$ *and* $\min(v) \geq \max(g)$*, then* $\dfrac{g}{x_{\max(g)}} v \in B$.

Proof: We consider two cases for the form of g.

Let $g = dx_{\max(g)}$. By Construction 3.4.3, it follows that $b = dx_{i+1}$. Hence, in this case $\dfrac{g}{x_{\max(g)}} v = dv = \dfrac{b}{x_{\max(b)}} v \in B$.

Let $g \neq dx_{\max(g)}$. Suppose that $\dfrac{g}{x_{\max(g)}} v \notin B$. Therefore, $\dfrac{b}{x_{\max(b)}} v \in \mathcal{C}$. This contradicts to the choice in Construction 3.4.3 that $bw \in \mathcal{C}$ is such that $b = \mathrm{begin}_l(\tilde{b})$ has minimal degree (since one can replace w by v). □

Lemma 3.4.5. *For each* $h < q$ *we have* $\mathrm{mingens}(B)_h = \mathrm{mingens}(N)_h$ *and* $B_h = N_h$.

Proof: The lemma holds because there are no gaps in B_h. □

Lemma 3.4.6. $\max(b) < \max(g)$.

Proof: We have that $\max(g) = i_r$ and $\max(b) = i + t$ by Construction 3.4.3. The argument in the proof of Lemma 3.3.2 yields that $\max(b) \leq \max(g)$. But $\max(b) = \max(g)$ contradicts to the choice in Construction 3.4.3 that $bw \in \mathcal{C}$ is such that $b = \mathrm{begin}_l(\tilde{b})$ has minimal degree (since one can replace w by $x_{\max(b)} w$). □

Lemma 3.4.7. *We denote by* $\mathrm{gb}(N)$ *the set of generators of* N *listed in the formulas in Construction 3.2.1.*

(i) If e *is a monomial of degree* q *such that* $e \succ_{lex} \tilde{g}$*, then* e *is divisible by a monomial in* $\mathrm{gb}(N)$*.*

(ii) If $gv \in B$ *is a* P*-free monomial with* $\min(v) \geq \max(g)$*, then the monomial* gv *is divisible by a monomial in* $\mathrm{gb}(N)$*.*

(iii) Let $bu \in \mathcal{C}$*. If* $h < \max(g)$*, then the monomial* $x_h gu$ *is divisible by a monomial in* $\mathrm{gb}(N)$*.*

Proof: (i) There are no gaps in B_q that are lex-greater than $\tilde{g}$. Therefore, $e \in B_q$. It follows that there exists a monomial $e' \in \mathrm{mingens}(B)$ that divides e. If $\deg(e') < q$, then $e' \in \mathrm{mingens}(N)$ by the previous lemma. If $\deg(e') = q$, then $e' \in \mathrm{mingens}(N)$ because there are no gaps lex-greater than e'.

(ii) Suppose that the monomial gv is not divisible by a monomial in $\mathrm{gb}(N)$. Since $gv \in B$, we have that gv is divisible by some minimal P-free monomial generator of B. As this generator is not in $\mathrm{gb}(N)$, it has to be an element in the set $\mathcal{C}$. Thus, there exists an u, such that $bu \in \mathcal{C}$ divides gv. By Lemma 3.4.6 it follows that the monomial b divides g, which is a contradiction.

(iii) Since $bu \in \mathcal{C}$, we have that the monomial gu is a gap in B. Therefore, $\deg(gu) \geq q$. Hence $\deg(x_h gu) > q$. Write $x_h gu = x_h gu'u''$ so that $\deg(x_h gu') = q$ and $\max(u') \leq \min(u'')$. We have that $x_h gu' \succ_{lex} \tilde{g}$. As $\tilde{g}$ is the lex-greatest gap in B_q, we can apply Lemma 3.4.7(i) and conclude that $x_h gu'$ is divisible by a monomial in $\mathrm{gb}(N)$. Hence, $x_h gu$ is divisible by a monomial in $\mathrm{gb}(N)$. □

3.5. Gröbner bases of N.

Lemma 3.5.1. *Denote by* $\prec$ *the degree-reverse-lexicographic order for which* $x_1 \prec x_2 \prec \ldots \prec x_n$*. The initial ideal* $\mathrm{in}_{\prec}(N)$ *contains* B*.*

Proof: Since $g \succ_{lex} b$, we have that $gu \succ_{lex} bu$ for each $bu \in \mathcal{C}$. Therefore, $bu \succ gu$ for each $bu \in \mathcal{C}$. Therefore, $\mathrm{in}_{\prec}(N) \supseteq B$. □

Construction 3.5.2. We will need a description of the first syzygies of B. Consider the following four types of syzygies.

(Syz 1) If $x_h^{a_h}$ and $x_f^{a_f}$ are minimal monomial generators of B, then there exists a first syzygy corresponding to the relation $x_h^{a_h} x_f^{a_f} - x_h^{a_h} x_f^{a_f} = 0$.

(Syz 2) Given a P-free monomial $a \in \text{mingens}(B)$ and a natural number $h < \max(a)$, let c be the pure-lex-greatest minimal monomial generator (of B) dividing $x_h a$. Sometimes we denote this by $c = \text{begin}(x_h a)$. Let z be the monomial such that $x_h a = zc$. We have a first syzygy corresponding to the relation $x_h a - zc = 0$. Note that $\min(z) \geq \max(c)$.

(Syz 3) Given a P-free monomial $a \in \text{mingens}(B)$ and a natural number $h < \max(a)$, such that x_h divides a and $x_h^{a_h} \in \text{mingens}(B)$, let x_h^f be the highest power of x_h that divides a. Set $z = \frac{a}{x_h^f}$. We have a first syzygy corresponding to the relation $x_h^{a_h - f} a - z x_h^{a_h} = 0$.

(Syz 4) Given a P-free monomial $a \in \text{mingens}(B)$ and a natural number h, such that x_h does not divide a and $x_h^{a_h} \in \text{mingens}(B)$, we have a first syzygy corresponding to the relation $x_h^{a_h} a - a x_h^{a_h} = 0$.

Lemma 3.5.3. *The set of all syzygies of the forms listed in Construction 3.5.2 contains a minimal system of generators of the first syzygy module of B.*

Proof: First, we make a remark about the pure powers contained in B. For each $1 \leq h \leq n$, denote by $\bar{a}_h \in \mathbf{N}$ the minimal power such that $x_h^{\bar{a}_h} \in B$. Clearly $\bar{a}_h \leq a_h$ since B contains P. It is possible that $x_h^{\bar{a}_h} \in \mathcal{C}$ if $\bar{a}_h < a_h$. Note that in this case $x_h^{\bar{a}_h}$ is a P-free monomial.

The ideal B is a monomial ideal, so Taylor's resolution provides a possibly non-minimal free resolution. The first syszygies of B in this resolution correspond to the relations of the form

$$\frac{\text{lcm}(m, m')}{m'} m' - \frac{\text{lcm}(m, m')}{m} m = 0\,,$$

where $m, m' \in \text{mingens}(B)$. If both m and m' are not P-free, then these are the syzygies of type (Syz 1). If one of m and m' is P-free and the other is not, then these are the syzygies of types (Syz 3) and (Syz 4). It remains to consider the case when both m and m' are P-free. We call such syzygies P-free syzygies, since the multidegree $\text{lcm}(m, m')$ of such a syzygy is P-free.

Denote by B' the monomial ideal generated by the P-free minimal monomial generators of B. By [GHP, Theorem 2.2], the minimal free resolution of B' is the P-free Eliahou-Kervaire resolution. Therefore, the syzygies of type (Syz 2) form a minimal set of generators of the first syzygy module of B'. By [GHP, Theorem 2.1] it follows that the syzygies of type (Syz 2) generate all P-free first sysyzgies of B. $\square$

Lemma 3.5.4. $\mathrm{in}_{\prec}(N) = B$.

Proof: We will prove that the set $\mathrm{gb}(N)$, defined in 3.4.7, is a Gröbner basis of the ideal N, defined in Construction 3.2.1. By [Ei, Theorem 15.8], it suffices to check that if $A, D \in \mathrm{gb}(N)$ and $\sigma\,\mathrm{in}(A) - \tau\,\mathrm{in}(D) = 0$ is a relation yielding a minimal first syzygy of B (where σ and τ are monomials), then $\sigma A - \tau D$ can be reduced to zero. By Lemma 3.5.3, it suffices to consider first syzygies of the four types listed in Construction 3.5.2. The case when both A and D are monomials is trivial. Suppose that A is a binomial. Then we have that $A = bu - gu$ for some $bu \in \mathcal{C}$. If D is a binomial, then we can write $D = bv - gv$ for some $bv \in \mathcal{C}$, and we get case (1) below. If D is a P-free monomial, then by Construction 3.5.2 (Syz 2) we can write $D = c$ for some $c \in \mathrm{gb}(N)$ and we get either case (2) or case (3) below. Let $D = x_h^{\bar{a}_h}$ for some $1 \leq h \leq n$. Then, by Construction 3.5.2 (Syz3 and Syz4) we get cases (4) and (5).

It follows that we have to check that each of the types of elements described below can be reduced to zero using elements in $\mathrm{gb}(N)$. Below e, c, u, v stand for monomials, and $bu \in \mathcal{C}$, $bv \in \mathcal{C}$. In particular, $bu \in \mathrm{mingens}(B)$ and $bv \in \mathrm{mingens}(B)$. Note that bu, $bv \in \mathrm{in}_{\prec}(N)$.

The five cases are:

(1) $e(bu - gu) - x_h(bv - gv)$,
where $ebu = x_h bv$, x_h divides bu, $h < \max(bv)$, and $\min(e) \geq \max(bu)$.

(2) $e(bu - gu) - x_h c$,
where $ebu = x_h c$, $c \in \mathrm{gb}(N)$, x_h divides bu, $h < \max(c)$, and $\min(e) \geq \max(bu)$; here $bu = \mathrm{begin}(x_h c)$.

(3) $x_h(bu - gu) - ec$,
where $x_h bu = ec$, $c \in \mathrm{gb}(N)$, x_h divides c, $h < \max(bu)$, and $\min(e) \geq \max(c)$; here $c = \mathrm{begin}(x_h bu)$.

(4) $x_h^{a_h - f}(bu - gu) - z x_h^{a_h}$,
where $h < \max(bu)$ is a natural number such that x_h divides bu, $x_h^{a_h} \notin \mathcal{C}$, x_h^f is the highest power of x_h that divides bu, and $z = \frac{bu}{x_h^f}$.

(5) $x_h^{a_h}(bu - gu) - bu x_h^{a_h}$,

where h is a natural number such that x_h does not divide bu and $x_h^{a_h} \notin \mathcal{C}$.

We consider each case separately.

(1) Consider the element

$$e(bu - gu) - x_h(bv - gv) = -egu + x_h gv\,.$$

Since $ebu = x_h bv$, it follows that $eu = x_h v$. Hence $-egu + x_h gv = 0$.

(2) Consider the element

$$e(bu - gu) - x_h c = -egu\,.$$

We have to show that the monomial egu is divisible by a monomial in $\mathrm{gb}(N)$. Suppose that egu is P-free; otherwise we are done.

If $\min(e) < \max(g)$, then by Lemma 3.4.7(iii) we have that the monomial egu is divisible by a monomial in $\mathrm{gb}(N)$. Suppose that $\min(e) \geq \max(g)$.

We consider two cases depending on whether x_h divides the monomial u.

First, we suppose that the variable x_h divides the monomial u. Set $v = \dfrac{u}{x_h}e$. Note that $\min(v) \geq \max(g)$ because $\min(e) \geq \max(g)$ and $\min(u) \geq \max(g)$ by Lemma 3.4.3. We have that $bv = c \in \mathrm{gb}(N)$. Since $bv \notin \mathcal{C}$ and $\min(v) \geq \max(g)$, it follows that $gv \in B$. By Lemma 3.4.7(ii), we get that the monomial gv is divisible by a monomial in $\mathrm{gb}(N)$. Thus, egu is divisible by a monomial in $\mathrm{gb}(N)$.

Now, we suppose that the variable x_h does not divide the monomial u. Therefore, x_h divides b. Set $v = ue$. We have that $\dfrac{b}{x_h}v = c \in \mathrm{mingens}(B)$. Since the variable x_h divides b, we have $h \leq \max(b)$. As $\dfrac{b}{x_h}v \in B$ and the ideal B is Borel+P, it follows that $\dfrac{b}{x_{\max(b)}}v \in B$.

By Lemmas 3.4.2 and 3.4.6 we have that $\min(u) \geq \max(g) > \max(b)$. Hence $\max(b) \leq \min(v)$. Therefore, there exists a $\dfrac{b}{x_{\max(b)}}v' \in \mathrm{mingens}(B)$ such that v' divides v and $\min(v') = \min(v)$. By Lemma 3.4.4 it follows that $\dfrac{g}{x_{\max(g)}}v' \in B$. Hence $gv \in B$.

By Lemma 3.4.7(ii), we get that the monomial gv is divisible by a monomial in $\mathrm{gb}(N)$. We conclude that egu is divisible by a monomial in $\mathrm{gb}(N)$.

(3) Consider the element

$$x_h(bu - gu) - ec = -x_h gu\,.$$

We have to show that the monomial $x_h gu$ is divisible by a monomial in $\mathrm{gb}(N)$. Suppose that $x_h gu$ is P-free; otherwise we are done.

If we have the inequality $h < \max(g)$, then the monomial $x_h gu$ is divisible by a monomial in $\mathrm{gb}(N)$ by Lemma 3.4.7(iii). Suppose that $h \geq \max(g)$ holds.

Since $x_h bu = ec$, we can write $c = \bar{b}\bar{u}x_h$, where $\bar{b}$ divides b and $\bar{u}$ divides u. Set $v = \bar{u}x_h$. We have that $\min(v) \geq \max(g)$. Suppose that $b \neq \bar{b}$. Then $\bar{b}\bar{u}x_h = c = \mathrm{begin}\big(x_h bu\big)$ implies that $h \leq \max(b)$. By Lemma 3.4.6, we get $h < \max(g)$, which is a contradiction. Therefore, $b = \bar{b}$. Then $bv = c \in \mathrm{gb}(N)$, so $bv \notin \mathcal{C}$ and $bv \in \mathrm{mingens}(B)$. Hence $gv \in B$, so $gvx_h \in B$, and then we can apply Lemma 3.4.7(ii).

(4) Consider the element

$$x_h^{a_h - f}(bu - gu) - zx_h^{a_h} = -x_h^{a_h - f} gu\,.$$

We consider two cases depending on whether the variable x_h divides the monomial b.

Suppose that the variable x_h divides the monomial b. Then the inequalities $h \leq \max(b) < \max(g)$ hold by Lemma 3.4.6. Since $bu \in \mathcal{C}$, we have that $bu \in \mathrm{mingens}(B)$, hence $x_h^{a_h}$ does not divide the monomial bu. Therefore, the variable x_h divides the monomial $x_h^{a_h - f}$. We can write $x_h^{a_h - f} gu = \frac{x_h^{a_h - f}}{x_h}(x_h gu)$. By Lemma 3.4.7(iii), it follows that the monomial $x_h gu$ is divisible by a monomial in $\mathrm{gb}(N)$. Hence, so is $x_h^{a_h - f} gu$.

Suppose that the variable x_h does not divide the monomial b. Therefore, x_h^f divides u. Hence $x_h^{a_h} \in N$ divides $x_h^{a_h - f} gu$.

(5) Note that $x_h^{a_h} \in \mathrm{mingens}(B)$ implies that $x_h^{a_h} \in \mathrm{gb}(N)$ by Construction 3.4.3. We have that the element

$$x_h^{a_h}(bu - gu) - bux_h^{a_h} = -x_h^{a_h} gu$$

is divisible by the monomial $x_h^{a_h} \in \mathrm{gb}(N)$.

The proof is finished since we have checked all cases. $\square$

3.6. Proof of Proposition 3.2.2. Recall by Construction 3.2.1 that $J = \mathrm{in}_{\prec_{lex}}(N)$. The following lemma follows from Construction 3.2.1.

Lemma 3.6.1.

(1) The monomial ideal J contains P.

(2) The ideal J is lex-closer than B.

Lemma 3.6.2. *The ideals J and B are connected by P-deformations and have the same Hilbert function.*

Proof: By Lemma 3.5.4 and Construction 3.2.1, we have that the ideals J and B are two different initial ideals of the binomial ideal N, and $N \supseteq P$. $\square$

The proof of Proposition 3.2.2 is complete.

3.7. Proof of Theorem 1.3. Now, we are ready to prove Theorem 1.3.

Theorem 1.3. *Every graded ideal in the polynomial ring S that contains the P-powers, is connected by a sequence of P-deformations to the lex+P ideal with the same Hilbert function.*

Proof: Let U be a graded ideal and $U \supseteq P$. Fix a monomial order $\succ$ in S. The initial ideal $J = \mathrm{in}_{\prec}(U)$ is a monomial+P ideal and is connected to U by a P-deformation.
Iteration step: If the monomial ideal J is not Borel+P, apply Proposition 3.1.5 to J. We obtain a Borel+P ideal B which is lex-closer than J. If B is not lex+P, apply Proposition 3.2.2. We obtain a new monomial+P ideal, which is lex-closer than B.

Apply repeatedly the iteration step above. At each step, we obtain an ideal which is lex-closer than the original monomial ideal. Since there exist only finitely many different monomial+P ideals with a fixed Hilbert function, it follows that the process terminates after finitely many steps. Therefore, the last ideal is lex+P.

We remark that the fact that there exist only finitely many different monomial+P ideals is obvious in the case $a_n < \infty$ when the Clements-Lindström ring is artinian. If $a_n = \infty$, then the fact follows from the Clements-Lindström Theorem [CL] since the theorem implies the following bound: if L is lex+P ideal and M is a monomial+P ideal with the same Hilbert function, then the maximal degree of a generator in mingens(L) is an upper bound for the degrees of the generators in mingens(M). $\square$

4. Maximal Betti numbers

In this section, we prove Theorem 1.5. The notation used throughout this section is disjoint from the notation used in Section 3.

First, we introduce special changes of coordinates and polarizations of a Borel+P ideal.

For a subset $\mathcal{A} \subset \{x_1, \ldots, x_n\}$ and for any monomial $m = x_1^{e_1} \cdots x_n^{e_n}$, let

$$\mathrm{pol}_{\mathcal{A}}(m) = \left(\prod_{x_j \in \mathcal{A},\ e_j \neq 0} (x_j y_{j,1} \cdots y_{j,e_j-1}) \right) \prod_{x_j \notin \mathcal{A}} x_j^{e_j},$$

where $y_{p,q}$ with $p, q \in \mathbf{N}$ are indeterminates. Let I be a monomial ideal of S. Write $\mathrm{pol}_{\mathcal{A}}(I)$ for the monomial ideal of $\tilde{S} = S[y_{p,q} : 1 \leq p \leq n,\ 1 \leq q \leq t]$ generated by $\{\mathrm{pol}_{\mathcal{A}}(u) : u \in \mathrm{mingens}(I)\}$, where t is a sufficiently large integer. Note that I and $\mathrm{pol}_{\mathcal{A}}(I)$ have the same graded Betti numbers, and therefore $I\tilde{S}$ and $\mathrm{pol}_{\mathcal{A}}(I)$ have the same Hilbert functions.

Definition 4.1. *Denote by $\mathcal{M}$ the set of all monomial+P ideals in S. Let $\mathcal{N} \subset \mathcal{M}$. An S-route φ of $\mathcal{N}$ is a map $\varphi : \mathcal{N} \to \mathcal{M}$ such that there exist a subset $\mathcal{A} \subset \{x_1, \ldots, x_n\}$, a linear transformation ϕ over $\tilde{S}$, and a monomial order $\prec$ on $\tilde{S}$ satisfying that, for each $I \in \mathcal{N}$, one has*

(A) $\mathrm{mingens}(\varphi(I)) = \mathrm{mingens}(\mathrm{in}_{\prec}(\phi(\mathrm{pol}_{\mathcal{A}}(I))))$,
(B) *$\varphi(I)$ is lex-closer than or equal to I.*

We simply say that φ is an S-route if $\mathcal{N} = \mathcal{M}$. The next proposition plays a crucial role in the proof of Theorem 1.5.

Proposition 4.2. *Let I be a Borel+P ideal of S which is not lex+P. For any finite set $\mathcal{N} \subset \mathcal{M}$ with $I \in \mathcal{N}$, there exists an S-route φ of $\mathcal{N}$ such that $\varphi(I) \neq I$.*

We will show that it is enough to prove the special case in Lemma 4.3.

Fix a variable x_j (here $1 \leq j \leq n$). Set $S(\hat{x}_j) = k[x_i : i \neq j]$ and $P(\hat{x}_j) = (\{x_i^{a_i} : i \neq j\}) \subset S(\hat{x}_j)$. We have that a monomial ideal L of $S(\hat{x}_j)$ is *lex+$P(\hat{x}_j)$* if L_d is lex-segment+$P(\hat{x}_j)$ for every integer $d \geq 0$.

A monomial+P ideal I decomposes (as a k-vector space) into a direct sum $I = \bigoplus_f fI_f$, where the sum runs over all monomials $f \in k[x_j]$. Each I_f is an ideal of

$S(\hat{x}_j)$ containing $P(\hat{x}_j)$. If all the ideals I_f are lex+$P(\hat{x}_j)$ ideals, we say that I is *x_j-compressed+P*.

We say that a monomial+P ideal I of S is *compressed+P* if I is x_j-compressed+P for every variable x_j. Note that compressed+P ideals are Borel+P if $n \geq 3$.

Lemma 4.3. *Suppose that $n \geq 3$. Let I be a compressed+P ideal of S which is not lex+P. For any finite set $\mathcal{N} \subset \mathcal{M}$ with $I \in \mathcal{N}$, there exists an S-route φ of $\mathcal{N}$ such that $\varphi(I) \neq I$.*

We first prove Proposition 4.2 by using Lemma 4.3.

Proof of Proposition 4.2: We use induction on n. If $n \leq 2$, then Borel+P ideals are lex+P. Suppose that $n \geq 3$. Let I be a monomial+P ideal in S and $\mathcal{N}$ a finite subset of $\mathcal{M}$ with $I \in \mathcal{N}$. If I is compressed+P, then the statement follows from Lemma 4.3.

Suppose that I is not compressed+P. Then there exists a variable x_j such that I is not x_j-compressed+P. For any monomial+P ideal $J \in \mathcal{N}$, consider the decomposition $J = \bigoplus_f f J_f$, where $f \in k[x_j]$ is a monomial and J_f is a monomial+$P(\hat{x}_j)$ ideal. Set $\mathcal{N}' = \{J_f : J \in \mathcal{N},\ f \in k[x_j]\}$. Since $J_{x_j^t} \subset J_{x_j^{t+1}}$ for $t = 0, 1, 2, \ldots$, the set $\{J_f : f \in k[x_j]\}$ is a finite set, and therefore $\mathcal{N}'$ is a finite set. We claim that, for any $S(\hat{x}_j)$-route φ of $\mathcal{N}'$, the map $J = \bigoplus_f f J_f \to \bigoplus_f f \varphi(J_f)$ is an S-route of $\mathcal{N}$.

Let $\tilde{S}(\hat{x}_j) = S(\hat{x}_j)[y_{p,q} : p \neq j] \subset \tilde{S}$. Then there exist $\mathcal{A} \subset \{x_1, \ldots, x_n\}$, a linear transformation ϕ over $\tilde{S}(\hat{x}_j)$ and a monomial order $\prec$ on $\tilde{S}(\hat{x}_j)$ such that $\mathrm{mingens}(\varphi(J_f))$ is equal to $\mathrm{mingens}(\mathrm{in}_\prec(\phi(\mathrm{pol}_\mathcal{A}(J_f))))$, where $\mathrm{pol}_\mathcal{A}(J_f)$ is an ideal of $\tilde{S}(\hat{x}_j)$. Consider the linear transformation $\tilde{\phi}$ over $\tilde{S}$ defined by $\tilde{\phi}(x_i) = \phi(x_i)$ and $\tilde{\phi}(y_{i,q}) = \phi(y_{i,q})$ if $i \neq j$ and $\tilde{\phi}(x_j) = x_j$ and $\tilde{\phi}(y_{j,q}) = y_{j,q}$. Let $\prec'$ be a monomial order on $\tilde{S}$ whose restriction to $\tilde{S}(\hat{x}_j)$ is the monomial order $\prec$. Then $\mathrm{in}_{\prec'}(\tilde{\phi}(\mathrm{pol}_\mathcal{A}(J))) = (\bigoplus_f f\varphi(J_f))\tilde{S}$. This fact shows that $\bigoplus_f f\varphi(J_f)$ is a monomial+P ideal and the map satisfies property (A) of S-routes. Also, the map satisfies property (B) since each $\varphi(J_f)$ is lex-closer than or equal to J_f.

Since each I_f is Borel+$P(\hat{x}_j)$, the induction hypothesis guarantees the existence of an $S(\hat{x}_j)$-route ψ of $\mathcal{N}'$ such that $I_m \neq \psi(I_m)$ for some monomial $m \in k[x_j]$. Then, we have $I \neq \bigoplus_f f\psi(I_f)$ as desired. $\square$

If $a_2 = \infty$ then lex+P ideals are lex ideals in the usual sense. Indeed, in this special case, Lemma 4.3 follows from the results in [Pa].

Lemma 4.4. [Pa] *Lemma 4.3 holds if $a_2 = \cdots = a_n = \infty$.*

We will prove the case $a_2 < \infty$ in a series of lemmas and constructions. More precisely, we show that if $a_2 < \infty$ then, for any compressed+P ideal I in S, there exists an S-route such that $\varphi(I) \neq I$ (we do not need to assume that $\mathcal{N}$ is finite).

4.5. Routes on S.
In the rest of this section except for Subsection 4.9, we assume $a_2 < \infty$. We introduce routes which will be used for the proof of Lemma 4.3.

Construction 4.5.1. Let $\zeta = \cos \frac{2\pi}{a_2} + \sqrt{-1} \sin \frac{2\pi}{a_2}$. Thus ζ is an a_2-th primitive root of unity.

Fix an integer $3 \leq r \leq n$.

Set

$$c = x_r + \cdots + x_n.$$

Let ϕ be the linear transformation of $\tilde{S}$ defined by

$$\phi(x_j) = \begin{cases} x_1 - \zeta c, & \text{if } j = 1, \\ x_1 - \zeta x_j, & \text{if } 2 \leq j \leq r-1, \\ x_j, & \text{if } j \geq r, \end{cases}$$

and

$$\phi(y_{i,j}) = \begin{cases} x_1 - \zeta^{j+1} c & + \ y_{1,j}, & \text{if } i = 1, \\ x_1 - \zeta^{-j} x_2 & + \ y_{2,j}, & \text{if } i = 2 \text{ and } 1 \leq j \leq a_2 - 2, \\ x_1 - \zeta^{j+1} x_i & + \ y_{i,j}, & \text{if } 2 < i \leq r-1 \text{ and } 1 \leq j \leq a_2 - 2, \\ x_1 - x_i & + \ y_{i,a_i-1}, & \text{if } 2 \leq i \leq r-1 \text{ and } j = a_i - 1, \\ x_i & + \ y_{i,j}, & \text{otherwise.} \end{cases}$$

Set $\mathbf{m}_Y = (\{y_{i,j} : 1 \leq i \leq n,\ 1 \leq j \leq t\}) \subset \tilde{S}$. We identify S and $\tilde{S}/\mathbf{m}_Y$ in a natural way. For any monomial $m = x_1^{e_1} \cdots x_n^{e_n}$ with $e_j \leq a_j$ for each j, let $\Phi(m)$ be the natural projection of $\phi(\mathrm{pol}_{\{x_1,\ldots,x_{r-1}\}}(m))$ to $\tilde{S}/\mathbf{m}_Y \simeq S$. Thus,

$$\Phi(x_1^{e_1} \cdots x_n^{e_n}) = \Phi(x_1^{e_1}) \cdots \Phi(x_n^{e_n})$$

and

$$
(4.0)\quad \Phi(x_j^{e_j}) = \begin{cases} \prod_{s=1}^{e_1}(x_1 - \zeta^s c), & \text{if } j = 1, \\ (x_1 - \zeta x_2)\left[\prod_{s=1}^{e_2-1}(x_1 - \zeta^{-s} x_2)\right], & \text{if } j = 2,\ 0 < e_2 < a_2, \\ \prod_{s=1}^{e_j}(x_1 - \zeta^s x_j), & \text{if } 2 < j \le r-1,\ e_j < a_2, \\ x_j^{e_j - a_2 + 1}\left[\prod_{s=1}^{a_2-1}(x_1 - \zeta^s x_j)\right], & \text{if } 2 < j \le r-1,\ a_2 \le e_j < a_j, \\ x_j^{e_j - a_2}\left[\prod_{s=1}^{a_2}(x_1 - \zeta^s x_j)\right], & \text{if } 2 \le j \le r-1,\ e_j = a_j, \\ x_j^{e_j}, & \text{otherwise.} \end{cases}
$$

Let I be a monomial+P ideal in S. We denote by $\Phi(I)$ the ideal of S generated by $\{\Phi(u) : u \in \text{mingens}(I)\}$. Fix a monomial order $\prec_Y$ on $S' = k[y_{i,j} : 1 \le i \le n,\ 1 \le j \le t]$. Let $\prec_B$ be the block monomial order on $\tilde{S}$ defined as follows: for monomials $uu', vv' \in \tilde{S}$, where $u, v \in S$ and $u', v' \in S'$, one has $uu' \succ_B vv'$ if $u \succ_{lex} v$, or $u = v$ and $u' \succ_Y v'$. Since $\prec_{lex}$ is the homogeneous lexicographic order, we have that

$$
\text{in}_{\prec_{lex}}\big(\Phi(I)\big) = \text{in}_{\prec_B}\big(\phi\big(\text{pol}_{\{x_1,\dots,x_{r-1}\}}(I)\big)\big) \cap S.
$$

We will prove the following result.

Proposition 4.5.2. *The map $I \to \text{in}_{\prec_{lex}}\Phi(I)$, constructed above, is an S-route.*

First, we prove that $\text{in}_{\prec_{lex}}\Phi(I)$ is a monomial+P ideal. Set

$$
\rho_j = \begin{cases} \prod_{s=1}^{a_1}(x_1 - \zeta^s c), & \text{if } j = 1, \\ x_j^{a_j - a_2}(c^{a_2} - x_j^{a_2}), & \text{if } 2 \le j \le r-1, \\ x_j^{a_j}, & \text{if } j \ge r. \end{cases}
$$

Note that the initial monomial of ρ_j is $x_j^{a_j}$ for all j.

Lemma 4.5.3. *The ideal $\Phi(P)$ is generated by the polynomials $\rho_1, \dots, \rho_n$. In particular, $\text{in}_{\prec_{lex}}\Phi(P) = P$.*

Proof: Clearly, $\rho_j = \Phi(x_j^{a_j}) \in \Phi(P)$ for $j = 1, r, r+1, \ldots, n$. On the other hand, $\Phi(x_j^{a_j}) = x_j^{a_j - a_2}(x_1^{a_2} - x_j^{a_2})$ for $2 \leq j \leq r-1$. Then the statement follows since

$$\rho_1 \left(\prod_{s=a_1+1}^{a_2} (x_1 - \zeta^s c) \right) = x_1^{a_2} - c^{a_2} \in \Phi(P)\,.$$

$\square$

For a set F of P-free monomials of degree d, we set

$$\mathcal{V}(F) = \mathrm{span}_k \big\{ \Phi(u) : u \in F \big\} \oplus_k \Phi(P)_d.$$

Lemma 4.5.4. *Let I be a monomial+P ideal of S and F_d the set of P-free monomials of degree d in I. Then $\mathcal{V}(F_d) \subset \Phi(I)_d$.*

Proof: Since t is a sufficiently large integer, for any P-free monomial $m \in F_d$, one has that $\mathrm{pol}_{\{x_1,\ldots,x_{r-1}\}}(m) \in \mathrm{pol}_{\{x_1,\ldots,x_{r-1}\}}(I)$, and therefore $\Phi(m) \in \Phi(I)$. Similarly, since $\mathrm{pol}_{\{x_1,\ldots,x_{r-1}\}}(P) \subset \mathrm{pol}_{\{x_1,\ldots,x_{r-1}\}}(I)$, we have the inclusion $\Phi(P)_d \subset \Phi(I)$. $\square$

For a polynomial $f(x_1, \ldots, x_n) \in S$ and for a linear form $\theta \in S_1$, denote

$$\mathrm{sub}(f; x_j, \theta) = f(x_1, \ldots, x_{j-1}, \theta, x_{j+1}, \ldots, x_n).$$

For an integer $\ell \in \mathbf{N}$ and a monomial $m = x_1^{e_1} \cdots x_n^{e_n}$ with $e_j \leq a_j$ for $j = 1, 2, \ldots, n$, define

$$\Phi(m : \zeta^\ell) = \Phi(x_1^{e_1} : \zeta^\ell) \cdots \Phi(x_n^{e_n} : \zeta^\ell)$$

and

$$\Phi(x_j^{e_j} : \zeta^\ell) = \begin{cases} \Phi(x_1^{e_1}), & \text{if } \ j = 1, \\ \mathrm{sub}\big(\Phi(x_j^{e_j}); x_1, \zeta^\ell c\big), & \text{if } \ j \geq 2. \end{cases}$$

Note that $\Phi(x_j^e : \zeta^\ell) = x_j^e$ if $j \geq r$.

Lemma 4.5.5. *Let $f \in S$ and $x_j - \theta \in S_1$ be a linear form. If $\mathrm{sub}(f; x_j, \theta) = 0$ then $f \in (x_j - \theta)$.*

Lemma 4.5.6. *For every P-free monomial $m \in S$, we have that $\mathrm{in}_{\prec_{lex}}\big(\Phi(m : \zeta^\ell)\big) = m$ for any $\ell \in \mathbf{N}$.*

A set L of P-free monomials of degree d is said to be a *P-free lex-segment* if the k-vector space $\mathrm{span}_k L \oplus_k P_d$ is lex-segment+P.

Lemma 4.5.7. *Let L be a P-free lex-segment set of P-free monomials of degree d. Then, for every monomial $m = x_1^{e_1} \cdots x_n^{e_n} \in L$, one has that the following properties hold.*
(i) $\Phi(x_1^{e_1+1})S_{d-e_1-1} \subset \mathcal{V}(L)$.
(ii) $\Phi(m : \zeta^{e_1+1}) \in \mathcal{V}(L)$.

Proof:
Step 1: First, we show that (i) implies (ii). Suppose that (i) holds. By definition,

$$\mathrm{sub}\big(\Phi(x_2^{e_2} \cdots x_n^{e_n}); x_1, \zeta^{e_1+1}c\big) = \Phi(x_2^{e_2} \cdots x_n^{e_n} : \zeta^{e_1+1}).$$

Recall that $\Phi(x_1^{e_1+1}) = \Phi(x_1^{e_1})(x_1 - \zeta^{e_1+1}c)$. By Lemma 4.5.5, we have

$$\Phi(x_1^{e_1}x_2^{e_2} \cdots x_n^{e_n}) - \Phi(x_1^{e_1}x_2^{e_2} \cdots x_n^{e_n} : \zeta^{e_1+1}) \in \Phi(x_1^{e_1+1})S_{d-e_1-1}.$$

Since $\Phi(x_1^{e_1} \cdots x_n^{e_n}) \in \mathcal{V}(L)$, we have $\Phi(x_1^{e_1} \cdots x_n^{e_n} : \zeta^{e_1+1}) \in \mathcal{V}(L)$ by (i).

Step 2: We prove (i) by using induction on $\#L$. If $\#L = 0$ then there is nothing to prove. Suppose that $\#L \geq 1$. Let $u = x_1^{e_1} \cdots x_n^{e_n}$ be the lex-smallest element in L. Set $L' = L \setminus \{u\}$. Since $\mathcal{V}(L) \supset \mathcal{V}(L')$, by the induction hypothesis, it is enough to prove the statement for u. If $u = x_1^d$ then there is nothing to prove. Thus, we may assume that $v \neq x_1^d$. Then, it is enough to show that for any monomial $w \in S_{d-e_1-1}$, there exists a polynomial $f_w \in \mathcal{V}(L)$ such that

$$\text{(4.1)} \qquad \Phi(x_1^{e_1+1}) \text{ divides } f_w \text{ and } \mathrm{in}_{\prec_{lex}}\left(\frac{f_w}{\Phi(x_1^{e_1+1})}\right) = \frac{\mathrm{in}_{\prec_{lex}}(f_w)}{x_1^{e_1+1}} = w.$$

We will consider two cases.

Case 1: Suppose that $x_1^{e_1+1}w$ is not a P-free monomial. Then some $x_t^{a_t}$ divides $x_1^{e_1+1}w$. The following polynomials satisfy (4.1):

- $\Phi(x_1^{a_1})\left(\dfrac{x_1^{e_1+1}w}{x_1^{a_1}}\right) \in \Phi(P)_d \subset \mathcal{V}(L)$, if $x_1^{a_1}$ divides $x_1^{e_1+1}w$,
- $\Phi(x_1^{e_1+1})\rho_t\left(\dfrac{w}{x_t^{a_t}}\right) \in \Phi(P)_d \subset \mathcal{V}(L)$, if $x_t^{a_t}$, where $t \neq 1$, divides $x_1^{e_1+1}w$.

Case 2: Suppose that $x_1^{e_1+1}w$ is a P-free monomial. Since $x_1^{e_1+1}w \succ_{lex} u$, we have $x_1^{e_1+1}w \in L'$. Then, by Step 1 (above) and Lemma 4.5.6, it follows that

$$\Phi(x_1^{e_1+1}w : \zeta^{e_1+2}) \in \mathcal{V}(L') \subset \mathcal{V}(L)$$

satisfies (4.1). This completes the proof. □

Corollary 4.5.8. *Let L be a P-free lex-segment set of P-free monomials of degree d and let $f_1, \ldots, f_t$ be a k-basis of $\Phi(P)_d$. Then*

(i) $\mathrm{in}_{\prec_{lex}} \mathcal{V}_r(L) = \mathrm{span}_k L \oplus_k P_d$*;*

(ii) $\{\Phi(u) : u \in L\} \cup \{f_1, \ldots, f_t\}$ *is a set of k-linearly independent polynomials.*

Proof: By the construction of $\mathcal{V}(L)$ and Lemma 4.5.3, we have that

$$\dim_k \mathcal{V}(L) \leq \#L + \dim_k \Phi(P)_d = \#L + \dim_k P_d,$$

and the equality holds if and only if (ii) holds. Hence, it is enough to show that $\mathrm{in}_{\prec_{lex}} \mathcal{V}(L) \supset \mathrm{span_k} L \oplus_k P_d$. We have the inclusion $\mathrm{in}_{\prec_{lex}} \mathcal{V}(L) \supset P_d$ by Lemma 4.5.3. On the other hand, for any monomial $m = x_1^{e_1} \cdots x_n^{e_n} \in L$, it follows from Lemmas 4.5.6 and 4.5.7(ii) that $\mathrm{in}_{\prec_{lex}} \Phi(m : \zeta^{e_1+1}) = m \in \mathrm{in}_{\prec_{lex}} \mathcal{V}(L)$. $\square$

Corollary 4.5.9. *Let F be a set of P-free monomials of degree d. Then the following properties hold.*

(i) $\dim_k \mathcal{V}(F) = \#F + \dim_k P_d$*.*

(ii) The set of P-free monomials in $\mathrm{in}_{\prec_{lex}} \mathcal{V}(F)$ *is lex-closer than or equal to F.*

Proof: (i) Let L be a P-free lex-segment set of P-free monomials of degree d with $L \supset F$, and let $f_1, \ldots, f_t$ be a k-basis of $\Phi(P)_d$. Corollary 4.5.8(ii) implies that the set $\{\Phi(u) : u \in F\} \cup \{f_1, \ldots, f_t\}$ is a set of k-linearly independent polynomials. By the construction of $\mathcal{V}(F)$, this fact implies $\dim_k \mathcal{V}(F) = \#F + \dim_k P_d$.

(ii) We use induction on $\#F$. If $\#F = 0$ then there is nothing to prove. Suppose that $\#F \geq 1$. Let u be the lex-smallest P-free monomial in F, and let $F' = F \setminus \{u\}$. Note that, by (i) and Lemma 4.5.3, the number of P-free monomials in $\mathrm{in}_{\prec_{lex}} \mathcal{V}(F)$ is equal to $\#F$. Consider the monomial $w \in \mathrm{in}_{\prec_{lex}} \mathcal{V}(F) \setminus \mathrm{in}_{\prec_{lex}} \mathcal{V}(F')$. It is enough to show that $w \succeq_{lex} u$. Let $L = \{v \in S_d : v \text{ is a } P\text{-free monomial with } v \succeq_{lex} u\ \}$. Then $w \in \mathrm{in}_{\prec_{lex}} \mathcal{V}(F) \subset \mathrm{in}_{\prec_{lex}} \mathcal{V}(L)$ and, by Corollary 4.5.8, the set of all P-free monomials in $\mathrm{in}_{\prec_{lex}} \mathcal{V}(L)$ is L. Since w is a P-free monomial, we have that $w \succeq_{lex} u$ as desired. $\square$

Lemma 4.5.10. *Let I be a monomial+P ideal of S and F_d the set of P-free monomials in I of degree d. Set* $\tilde{J} = \mathrm{in}_{\prec_B} \phi(\mathrm{pol}_{\{x_1,\ldots,x_{r-1}\}}(I))$ *and $J = \tilde{J} \cap S$. Then*

(i) $J_d = \mathrm{in}_{\prec_{lex}} \mathcal{V}(F_d)$*.*

(ii) The ideals I and J have the same Hilbert function.

(iii) The ideals $\tilde{J}$ and J have the same generators.

Proof: Since $\mathcal{V}(F_d) \subset \Phi(I)_d$ and $J = \text{in}_{\prec_{lex}} \Phi(I)$, it follows from Corollary 4.5.9 that

$$\text{Hilb}(J)(d) \geq \dim_k \mathcal{V}(F_d) = \#F_d + \dim_k P_d = \text{Hilb}(I)(d) \quad \text{for all } d.$$

On the other hand, since $\tilde{J} \supset J\tilde{S}$, the above inequality implies

$$\text{Hilb}(\tilde{J})(d) \geq \text{Hilb}(J\tilde{S})(d) \geq \text{Hilb}(I\tilde{S})(d) = \text{Hilb}\big(\text{pol}_{\{x_1,\dots,x_{r-1}\}}(I)\big)(d) = \text{Hilb}(\tilde{J})(d)$$

for all $d \geq 0$. Thus all of the above Hilbert functions are the same. In particular, $\text{Hilb}(\tilde{J}) = \text{Hilb}(J\tilde{S})$ and $\text{Hilb}(J) = \text{Hilb}(I)$. Since $\tilde{J} \supset J\tilde{S}$, this proves (ii) and (iii).

Finally, since $\text{in}_{\prec_{lex}} \mathcal{V}(F_d) \subset J_d$ and since

$$\dim_k \big(\text{in}_{\prec_{lex}} \big(\mathcal{V}(F_d)\big)\big) = \#F_d + \dim_k P_d = \dim_k I_d = \dim_k J_d,$$

it follows that $\text{in}_{\prec_{lex}} \mathcal{V}(F_d) = J_d$. □

Now, we will show Proposition 4.5.2.

Proof of Proposition 4.5.2: It follows from Lemmas 4.5.3 and 4.5.4 that $\text{in}_{\prec_{lex}} \Phi(I)$ contains P. Also, since $\text{in}_{\prec_{lex}} \Phi(I) = \text{in}_{\prec_B} \phi(\text{pol}_{\{x_1,\dots,x_{r-1}\}}(I)) \cap S$, property (A) and (B) of S-routes follow from Corollary 4.5.9(ii) and Lemma 4.5.10(iii). □

4.6. Proof of Lemma 4.3.

First, we remark the following obvious fact.

Lemma 4.6.1. *Let $u = x_1^{c_1} \cdots x_n^{c_n}$ and $v = x_1^{e_1} \cdots x_n^{e_n}$ be P-free monomials of the same degree with $u \succ_{lex} v$ and $v \in I$. If $c_j = e_j$ for some j then $u \in I$.*

We will prove Lemma 4.3 by using the route defined in Construction 4.5.1. Let I be a compressed+P ideal which is not lex+P and let q be the smallest integer d such that I_d is not lex-segment+P. Let g be the lex-greatest gap of I_q and $\alpha_1 = \max\{j \in \mathbf{N} : x_1^j \text{ divides } g\}$. Let $\tilde{g} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be the lex-smallest P-free monomial of degree q which is divisible by $x_1^{\alpha_1}$ and $\tilde{b} = x_1^{\beta_1} \cdots x_n^{\beta_n}$ the lex-greatest P-free monomial in I_q which is lex-smaller than g.

Lemma 4.6.2. *$\beta_1 = \alpha_1 - 1$ and $\tilde{g} \notin I$.*

Proof: Since $\tilde{b} \prec_{lex} g$, we have the inequality $\beta_1 \leq \alpha_1$. However, $\beta_1 \neq \alpha_1$ by Lemma 4.6.1. Then $\beta_1 = \alpha_1 - 1$ since I is Borel+P. Furthermore, $\tilde{g} \notin I$ follows from Lemma 4.6.1. □

Let F_q be the set of all P-free monomials in I_q. Set

$$\tilde{L} = \{u \in S_q : u \text{ is a } P\text{-free monomial with } u \succ_{lex} \tilde{g}\}$$

and

$$G = F_q \cup \tilde{L}.$$

Lemma 4.6.3. *If $\tilde{g} \in \mathrm{in}_{\prec_{lex}} \mathcal{V}(G)$, then $\mathrm{in}_{\prec_{lex}} \mathcal{V}(F_q) \neq I_q$.*

Proof: Let $t = \#\{u \in F_q : \ u \text{ is not divisible by } x_1^{\alpha_1}\}$. By the assumption and Corollary 4.5.8, $\mathrm{in}_{\prec_{lex}} \mathcal{V}(G)$ contains all monomials of degree q which is divisible by $x_1^{\alpha_1}$. Hence

$$\#\{u \in \mathrm{in}_{\prec_{lex}} \mathcal{V}(G) : \ u \text{ is not divisible by } x_1^{\alpha_1}\} = t - 1.$$

As $\mathcal{V}(F_q) \subset \mathcal{V}(G)$, we have that

$$\#\{u \in \mathrm{in}_{\prec_{lex}} \mathcal{V}(F_q) : \ u \ \text{ is not divisible by } x_1^{\alpha_1}\} \leq t - 1.$$

Hence $\mathrm{in}_{\prec_{lex}} \mathcal{V}(F_q) \neq I_q$. $\square$

Recall that what we need to prove is that there exists an r for which $\mathrm{in}_{\prec_{lex}} \Phi(I) \neq I$. Also, $\mathrm{in}_{\prec_{lex}} \Phi(I)_q$ is equal to $\mathrm{in}_{\prec_{lex}} \mathcal{V}(F_q)$ by Lemma 4.5.10(i). By Lemma 4.6.3 it follows that the next lemma completes the proof of Lemma 4.3.

Lemma 4.6.4. *There exists an $2 \leq r \leq n$ so that $\tilde{g} \in \mathrm{in}_{\prec_{lex}} \mathcal{V}(G)$.*

The proof of Lemma 4.6.4 consists of considering two cases: when $n = 3$ and when $n > 3$. These cases are considered in Subsections 4.7 and 4.8 respectively.

4.7. Proof of Lemma 4.6.4 when $n = 3$

Throughout this subsection, we suppose that $n = 3$ and $r = 3$. We will show that

$$\tilde{g} \in \mathrm{in}_{\prec_{lex}} \mathcal{V}(G).$$

Let $\tilde{g} = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ and $\tilde{b} = x_1^{\alpha_1 - 1} x_2^{\beta_2} x_3^{\beta_3}$. Note that $\tilde{g} \notin G$, $\tilde{b} \in G$, $\alpha_2 < \beta_2$, $\alpha_3 > \beta_3$ and $c = x_3$. Since $x_1^{\alpha_1} x_2^{\alpha_2 + 1} x_3^{\alpha_3 - 1} \in \tilde{L} \subset G$, by Lemma 4.5.7(i), we have

$$\Phi(x_1^{\alpha_1 + 1}) S_{q - \alpha_1 - 1} \subset \mathcal{V}(G). \tag{4.2}$$

For $t = 0, 1, \dots, \beta_2 - 2$, let

$$f_t = \Phi(x_1^{\alpha_1})x_3^{\beta_3}(x_1 - \zeta x_2)\left[\prod_{s=1}^{t}(x_1 - \zeta^{-s}x_2)\right]\left[\prod_{s=t+1}^{\beta_2-2}(\zeta^{\alpha_1+1}x_3 - \zeta^{-s}x_2)\right]$$

Lemma 4.7.1. *$f_t \in \mathcal{V}(G)$ for $t = 0, 1, \dots, \beta_2 - 2$.*

Proof: Since I is Borel+P, the monomial $x_1^{\alpha_1}x_2^{\beta_2-1}x_3^{\beta_3} \in \tilde{L}$. Then, by Lemma 4.5.7(ii), we have $\Phi(x_1^{\alpha_1}x_2^{\beta_2-1}x_3^{\beta_3} : \zeta^{\alpha_1+1}) \in \mathcal{V}(G)$. On the other hand, for $t = 0, 1, 2 \dots, \beta_2 - 2$, one has

$$\mathrm{sub}\left(\frac{f_t}{\Phi(x_1^{\alpha_1})}; x_1, \zeta^{\alpha_1+1}x_3\right) = \Phi(x_2^{\beta_2-1}x_3^{\beta_3} : \zeta^{\alpha_1+1}).$$

Then, by Lemma 4.5.5,

$$f_t - \Phi(x_1^{\alpha_1}x_2^{\beta_2-1}x_3^{\beta_3} : \zeta^{\alpha_1+1}) \in \Phi(x_1^{\alpha_1})(x_1 - \zeta^{\alpha_1+1}x_3)S_{q-\alpha_1-1} = \Phi(x_1^{\alpha_1+1})S_{q-\alpha_1-1}.$$

Thus, by (4.2), we have $f_t \in \mathcal{V}(G)$ for $t = 0, 1, \dots, \beta_2 - 2$. □

For $t = 0, 1, \dots, \beta_2 - 1$, let

$$h_t = \Phi(x_1^{\alpha_1-1})x_3^{\beta_3}(x_1 - \zeta x_2)\left[\prod_{s=1}^{t}(x_1 - \zeta^{-s}x_2)\right]\left[\prod_{s=t}^{\beta_2-2}(\zeta^{\alpha_1+1}x_3 - \zeta^{-s}x_2)\right].$$

Lemma 4.7.2. *$h_t \in \mathcal{V}(G)$ for $t = 0, 1, \dots, \beta_2 - 1$.*

Proof: For $t = 1, 2, \dots, \beta_2 - 1$, one has

$$\begin{aligned}
h_t - f_{t-1} &= \Phi(x_1^{\alpha_1-1})x_3^{\beta_3}(x_1 - \zeta x_2)\left[\prod_{s=1}^{t-1}(x_1 - \zeta^{-s}x_2)\right]\left[\prod_{s=t}^{\beta_2-2}(\zeta^{\alpha_1+1}x_3 - \zeta^{-s}x_2)\right] \\
&\quad \times (x_1 - \zeta^{-t}x_2 - x_1 + \zeta^{\alpha_1}x_3) \\
&= \Phi(x_1^{\alpha_1-1})x_3^{\beta_3}(x_1 - \zeta x_2)\left[\prod_{s=1}^{t-1}(x_1 - \zeta^{-s}x_2)\right]\left[\prod_{s=t}^{\beta_2-2}(\zeta^{\alpha_1+1}x_3 - \zeta^{-s}x_2)\right] \\
&\quad \times \zeta^{-1}(\zeta^{\alpha_1+1}x_3 - \zeta^{-t+1}x_2) \\
&= \zeta^{-1}h_{t-1}.
\end{aligned}$$

Since each f_t is in $\mathcal{V}(G)$ and $h_{\beta_2-1} = \Phi(\tilde{b}) \in \mathcal{V}(G)$, the above equality implies that $h_t \in \mathcal{V}(G)$ for $t = 0, 1, \ldots, \beta_2 - 1$. $\square$

Now, since $(x_1 - \zeta x_2) = (x_1 - \zeta^{\alpha_1} x_3) + (\zeta^{\alpha_1} x_3 - \zeta x_2)$, it follows that $h_0 \in \mathcal{V}(G)$ can be written in the form

$$h_0 = \Phi(x_1^{\alpha_1}) x_3^{\beta_3} \left[\prod_{s=0}^{\beta_2-2} (\zeta^{\alpha_1+1} x_3 - \zeta^{-s} x_2) \right] + f', \tag{4.3}$$

where $\mathrm{in}_{\prec_{lex}}(f') = x_1^{\alpha_1-1} x_2^{\beta_2} x_3^{\beta_3} \prec_{lex} \tilde{g}$. Let

$$\tilde{h}_t = \Phi(x_1^{\alpha_1}) x_3^{\beta_3+t+1} \left[\prod_{s=1}^{\beta_2-2-t} (\zeta^{\alpha_1+1} x_3 - \zeta^{-s} x_2) \right] \quad \text{for } t = 0, 1, \ldots, \alpha_3 - \beta_3 - 1.$$

Since $\mathrm{in}_{\prec_{lex}}(\tilde{h}_{\alpha_3-\beta_3-1}) = \tilde{g}$, the next lemma completes the proof of Lemma 4.6.4.

Lemma 4.7.3. *There exists a number $\delta \in k \setminus \{0\}$ such that $\delta \tilde{h}_{\alpha_3-\beta_3-1} + f' \in \mathcal{V}(G)$.*

Proof: For $t = 0, 1, \ldots, \alpha_3 - \beta_3 - 1$, we have $x_1^{\alpha_1} x_2^{\beta_2-1-t} x_3^{\beta_3+t} \succ_{lex} \tilde{g}$, and therefore $x_1^{\alpha_1} x_2^{\beta_2-1-t} x_3^{\beta_3+t} \in \tilde{L}$. Thus, by Lemma 4.5.7(ii), we get $\Phi(x_1^{\alpha_1} x_2^{\beta_2-1-t} x_3^{\beta_3+t} : \zeta^{\alpha_1+1}) \in \mathcal{V}(G)$ for $t = 0, 1, \ldots, \alpha_3 - \beta_3 - 1$. Then we have

$$\zeta^{\alpha_1}(\zeta - 1)\tilde{h}_0 + f' \in \mathcal{V}(G)$$

by using (4.3) and the following computation:

$$\begin{aligned}
&\Phi(x_1^{\alpha_1}) x_3^{\beta_3} \left[\prod_{s=0}^{\beta_2-2} (\zeta^{\alpha_1+1} x_3 - \zeta^{-s} x_2) \right] - \zeta^{-1} \Phi(x_1^{\alpha_1} x_2^{\beta_2-1} x_3^{\beta_3} : \zeta^{\alpha_1+1}) \\
&= \Phi(x_1^{\alpha_1}) x_3^{\beta_3} \left[\prod_{s=1}^{\beta_2-2} (\zeta^{\alpha_1+1} x_3 - \zeta^{-s} x_2) \right] \left\{ \zeta^{\alpha_1+1} x_3 - x_2 - \zeta^{-1} (\zeta^{\alpha_1+1} x_3 - \zeta x_2) \right\} \\
&= \zeta^{\alpha_1}(\zeta - 1)\tilde{h}_0.
\end{aligned}$$

If $\alpha_3 - \beta_3 - 1 = 0$, then this completes the proof.

If $\alpha_3 - \beta_3 - 1 > 0$, then the statement follows from the next computation. For $t = 0, 1, \ldots, \alpha_3 - \beta_3 - 2$, we get

$$\begin{aligned}
&\tilde{h}_t - \zeta^{-\beta_2+1+t}\Phi_3(x_1^{\alpha_1}x_2^{\beta_2-1-(t+1)}x_3^{\beta_3+t+1} : \zeta^{\alpha_1+1}) \\
&= \Phi(x_1^{\alpha_1})x_3^{\beta_3+t+1}\left[\prod_{s=1}^{\beta_2-2-(t+1)}(\zeta^{\alpha_1+1}x_3 - \zeta^{-s}x_2)\right] \\
&\quad \times \left\{\zeta^{\alpha_1+1}x_3 - \zeta^{-\beta_2+2+t}x_2 - \zeta^{-\beta_2+1+t}(\zeta^{\alpha_1+1}x_3 - \zeta x_2)\right\} \\
&= \zeta^{\alpha_1+1}(1 - \zeta^{t+1-\beta_2})\tilde{h}_{t+1}.
\end{aligned}$$

Note that $\zeta^{t+1-\beta_2} \neq 1$ since $1 - \beta_2 \leq t + 1 - \beta_2 \leq \alpha_3 - \beta_3 - \beta_2 - 1 = -\alpha_2 - 2$ and since $-a_2 < 1 - \beta_2 \leq -\alpha_2 - 2 < 0$. □

4.8. Proof of Lemma 4.6.4 when $n \geq 4$

In this subsection we consider the case $n \geq 4$.

By the definition of $\tilde{g}$ and $\tilde{b}$, the monomials $\tilde{g}$ and $\tilde{b}$ can be written in the form

$$\tilde{g} = x_1^{\alpha_1}x_p^{\alpha_p}x_{p+1}^{a_{p+1}-1}\cdots x_n^{a_n-1} \qquad \text{with } 2 \leq p \leq n \text{ and } \alpha_p > 0$$

and

$$\tilde{b} = x_1^{\alpha_1-1}x_2^{a_2-1}\cdots x_{\ell-1}^{a_{\ell-1}-1}x_\ell^{\beta_\ell} \qquad \text{with } 2 \leq \ell \leq n \text{ and } \beta_\ell < a_\ell - 1.$$

For convenience, we will write $\tilde{b} = x_1^{\alpha_1-1}x_2^{\beta_2}\cdots x_n^{\beta_n}$.

Lemma 4.8.1.

(i) $p \geq 3$.

(ii) The monomial $\tilde{b}$ satisfies one of the following conditions:

(1) $\tilde{b} = x_1^{\alpha_1-1}x_2^{\beta_2}\cdots x_p^{\beta_p}$ *and* $0 \leq \beta_p < \alpha_p$;

(2) $\tilde{b} = x_1^{\alpha_1-1}x_2^{\beta_2}\cdots x_p^{\beta_p}x_{p+1}^{\beta_{p+1}}$, $\beta_p > \alpha_p$ *and* $0 \leq \beta_{p+1} < a_{p+1} - 1$.

Proof: Statement (ii) easily follows from Lemma 4.6.1. Suppose that $p = 2$. Then $\tilde{g} = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{a_3-1}\cdots x_n^{a_n-1}$. Since $n \geq 4$, we get $\deg\, \tilde{g} \geq \alpha_1 - 1 + a_2 - 1 + a_3 - 1$. Therefore, $\tilde{b}$ is divisible by $x_1^{\alpha_1-1}x_2^{a_2-1}x_3^{a_3-1}$. In particular $\beta_3 = a_3 - 1$. By Lemma 4.6.1, it follows that $\tilde{g} \in I$, which is a contradiction. □

Let

$$r = \begin{cases} p, & \text{if } \tilde{b} \text{ is a monomial of the form (1),} \\ p+1, & \text{if } \tilde{b} \text{ is a monomial of the form (2).} \end{cases}$$

Our goal is to prove $\tilde{g} \in \mathrm{in}_{\prec_{lex}}(\mathcal{V}(G))$.

Lemma 4.8.2.
(i) $\Phi(x_1^{\alpha_1} x_p^{\alpha_p+1} : \zeta^{\alpha_1+1}) S_{q-\alpha_1-\alpha_p-1} \subset \mathcal{V}(G)$ *and* $\Phi(x_1^{\alpha_1} x_j : \zeta^{\alpha_1+1}) S_{q-\alpha_1-1} \subset \mathcal{V}(G)$ *for* $j = 1, 2, \ldots, p-1$;
(ii) $\Phi(x_1^{\alpha_1} x_2) S_{q-\alpha_1-1} \subset \mathcal{V}(G)$.

Proof: (i) Let $u_j = x_1^{\alpha_1} x_j$ for $j = 1, 2, \ldots, p-1$ and $u_p = x_1^{\alpha_1} x_p^{\alpha_p+1}$. Let $d_j = \deg u_j$. Since $\frac{\tilde{g} x_2}{x_n} \in \tilde{L}$ and $\tilde{L}$ is P-free lex-segment, it follows from Lemma 4.5.7(i) that $\Phi(u_1 : \zeta^{\alpha_1+1}) S_{q-\alpha_1-1} = \Phi(x_1^{\alpha_1+1}) S_{q-\alpha_1-1} \subset \mathcal{V}(G)$. Fix a $2 \leq j \leq p$. In the same way as in the proof of Lemma 4.5.7, it is enough to show that, for every monomial $w \in S_{q-d_j}$, there exists a polynomial $f_w \in \mathcal{V}(G)$ such that

$$\Phi(u_j : \zeta^{\alpha_1+1}) \text{ divides } f_w \text{ and } \frac{\mathrm{in}_{\prec_{lex}}(f_w)}{u_j} = w. \tag{4.4}$$

We will consider two cases.

Case 1: Suppose that $u_j w$ is not a P-free monomial. Then one of the following polynomials satisfies (4.4):

$$(a) \quad \rho_1 \Phi\left(\frac{u_j}{x_1^{\alpha_1}} : \zeta^{\alpha_1+1}\right)\left(\frac{w x_1^{\alpha_1}}{x_1^{a_1}}\right) \in \Phi(P)_q, \quad \text{where } x_1^{a_1} \text{ divides } u_j w,$$

$$(b) \quad \Phi(x_1^{\alpha_1}) \rho_j \left(\frac{u_j w}{x_1^{\alpha_1} x_j^{a_j}}\right) \in \Phi(P)_q, \quad \text{where } x_j^{a_j} \text{ divides } u_j w,$$

$$(c) \quad \rho_t \Phi(u_j : \zeta^{\alpha_1+1}) \left(\frac{w}{x_t^{a_t}}\right) \in \Phi(P)_q, \quad \text{where } x_t^{a_t} \text{ divides } u_j w \text{ and } t \neq 1, j.$$

Note that $\Phi(u_j : \zeta^{\alpha_1+1})$ divides the polynomial (b) since

$$\rho_j = x_j^{a_j - a_2} \{ (\zeta^{\alpha_1+1} c)^{a_2} - x_j^{a_j} \} = x_j^{a_j - a_2} \left[\prod_{s=1}^{a_2} (\zeta^{\alpha_1+1} c - \zeta^s x_j) \right].$$

Case 2: Suppose that $u_j w = x_1^{e_1} \cdots x_n^{e_n}$ is a P-free monomial. If $e_1 > \alpha_1$, then since (i) holds for $j = 1$, we get that

$$\Phi(u_j x_1 : \zeta^{\alpha_1+1}) \left(\frac{w}{x_1} \right) \in \Phi(x_1^{\alpha_1+1}) S_{q-\alpha_1-1} \subset \mathcal{V}(G)$$

satisfies (4.4). Suppose $e_1 = \alpha_1$. Then $u_j w \in \tilde{L}$ since $u_j w \succ_{lex} \tilde{g}$. Then it follows from Lemmas 4.5.6 and 4.5.7 that $\Phi(u_j w : \zeta^{\alpha_1+1}) \in \mathcal{V}(G)$ satisfies (4.4). We proved (i).

(ii) Since $p \geq 3$, we have the inclusions $\Phi(x_1^{\alpha_1} x_2 : \zeta^{\alpha_1+1}) S_{q-\alpha_1-1} \subset \mathcal{V}(G)$ and $\Phi(x_1^{\alpha_1+1}) S_{q-\alpha_1-1} \subset \mathcal{V}(G)$. The statement follows since $\Phi(x_1^{\alpha_1} x_2) = \Phi(x_1^{\alpha_1} x_2 : \zeta^{\alpha_1+1}) + \Phi(x_1^{\alpha_1+1})$. □

Lemma 4.8.3. *There exists a polynomial f' such that $\mathrm{in}_{\prec_{lex}}(f') = \tilde{b}$ and*

$$\Phi(x_1^{\alpha_1}) \Phi(x_p^{\beta_p} \cdots x_n^{\beta_n} : \zeta^{\alpha_1}) c^{\beta_2+\cdots+\beta_{p-1}-1} + f' \in \mathcal{V}(G).$$

Proof: Let

$$\Gamma = \frac{\Phi(x_2^{\beta_2} \cdots x_n^{\beta_n} : \zeta^{\alpha_1})}{(\zeta^{\alpha_1} c - x_2)}.$$

Since

$$\mathrm{sub}\left(\frac{\Phi(\tilde{b})}{\Phi(x_1^{\alpha_1-1} x_2)}; x_1, \zeta^{\alpha_1} c \right) = \Gamma,$$

it follows from Lemma 4.5.5 that

$$\Phi(\tilde{b}) - \Phi(x_1^{\alpha_1-1} x_2)\Gamma \in \Phi(x_1^{\alpha_1-1} x_2)(x_1 - \zeta^{\alpha_1} c) S_{q-\alpha_1-1} = \Phi(x_1^{\alpha_1} x_2) S_{q-\alpha_1-1}.$$

Then $\Phi(x_1^{\alpha_1-1} x_2)\Gamma \in \mathcal{V}(G)$ by Lemma 4.8.2(ii). As $\Phi(x_1^{\alpha_1-1} x_2) = \Phi(x_1^{\alpha_1}) + \Phi(x_1^{\alpha_1-1} x_2 : \zeta^{\alpha_1})$, we have

$$\Phi(x_1^{\alpha_1})\Gamma + \Phi(x_1^{\alpha_1-1} x_2 : \zeta^{\alpha_1})\Gamma \in \mathcal{V}(G). \tag{4.5}$$

Let

$$f_t = \Phi(x_1^{\alpha_1}) \Phi(x_t^{\beta_t} \cdots x_n^{\beta_n} : \zeta^{\alpha_1}) c^{\beta_2+\cdots+\beta_{t-1}-1} \quad \text{for } t = 3, 4, \ldots, p.$$

Note that $\Phi(x_1^{\alpha_1}x_j : \zeta^{\alpha_1+1}) = \zeta(\zeta^{\alpha_1}c - x_j)\Phi(x_1^{\alpha_1})$ for $j = 2, \ldots, p-1$. By (4.0), there exists a number $\delta_2 \in k \setminus \{0\}$ such that

$$\mathrm{sub}(\Gamma; x_2, \zeta^{\alpha_1}c) = \delta_2 c^{\beta_2 - 1}\Phi(x_3^{\beta_3}\cdots x_n^{\beta_n} : \zeta^{\alpha_1}).$$

Then, by Lemmas 4.5.5 and 4.8.2(i), we obtain

$$\Phi(x_1^{\alpha_1})\Gamma - \delta_2 f_3 \in \Phi(x_1^{\alpha_1}x_2 : \zeta^{\alpha_1+1})S_{q-\alpha_1-1} \subset \mathcal{V}(G). \tag{4.6}$$

Similarly, for $t = 3, 4, \ldots, p-1$, it follows from (4.0) that there exists a number $\delta_t \in k\setminus\{0\}$ such that

$$\mathrm{sub}\big(\Phi(x_t^{\beta_t}\cdots x_n^{\beta_n} : \zeta^{\alpha_1}); x_t, \zeta^{\alpha_1}c\big) = \delta_t c^{\beta_t}\Phi(x_{t+1}^{\beta_{t+1}}\cdots x_n^{\beta_n} : \zeta^{\alpha_1}).$$

Therefore,

$$f_t - \delta_t f_{t+1} \in \Phi(x_1^{\alpha_1}x_t : \zeta^{\alpha_1+1})S_{q-\alpha_1-1} \subset \mathcal{V}(G) \quad \text{for } t = 3, 4, \ldots, p-1. \tag{4.7}$$

Now, (4.5), (4.6) and (4.7) imply that

$$(\delta_2\cdots\delta_{p-1})f_p + \Phi(x_1^{\alpha_1-1}x_2 : \zeta^{\alpha_1})\Gamma \in \mathcal{V}(G).$$

The lemma follows since $(\delta_2\cdots\delta_{p-1}) \in k \setminus \{0\}$ and $\mathrm{in}_{\prec_{lex}}(\Phi(x_1^{\alpha_1-1}x_2 : \zeta^{\alpha_1})\Gamma) = x_1^{\alpha_1-1}x_2^{\beta_2}\cdots x_n^{\beta_n} = \tilde{b}$. □

Lemma 4.8.4. *There exists a polynomial h such that $\mathrm{in}_{\prec_{lex}}(h) \prec_{lex} \tilde{g}$ and*

$$\Phi(x_1^{\alpha_1}x_{r-1}^{\alpha_{r-1}} : \zeta^{\alpha_1+1})x_r^{\beta_r}c^{q-\alpha_1-\alpha_{r-1}-\beta_r} + h \in \mathcal{V}(G),$$

where $\alpha_{r-1} = 0$ if $r = p$.

Proof: Recall that, for all $\ell, e \in \mathbf{N}$, $\Phi(x_j^e : \zeta^\ell) = x_j^e$ if $j \geq r$. If $\tilde{b}$ is a monomial of the form (1), then the statement is exactly Lemma 4.8.3. Suppose $\tilde{b}$ is a monomial of the form (2). By Lemma 4.8.3, there exists a polynomial f' with $\mathrm{in}_{\prec_{lex}}(f') = \tilde{b}$ such that

$$\Phi(x_1^{\alpha_1})\Phi(x_p^{\beta_p} : \zeta^{\alpha_1})x_{p+1}^{\beta_{p+1}}c^{q-\alpha_1-\beta_p-\beta_{p+1}} + f' \in \mathcal{V}(G). \tag{4.8}$$

Note that $0 < \alpha_p < \beta_p$. Let

$$\tau = \frac{\Phi(x_p^{\alpha_p+1} : \zeta^{\alpha_1+1})}{(\zeta^{\alpha_1+1}c - \zeta x_p)}.$$

We will need the following:

Claim.

(i) τ divides $\Phi(x_p^{\beta_p} : \zeta^{\alpha_1})$.

(ii) There exist a number $\delta \in k \setminus \{0\}$ and $f'' \in S$, such that $\mathrm{in}_{\prec_{lex}}(f'') = x_p^{\alpha_p - 1}x_{p+1}$ and

$$\tau = \delta\Phi(x_p^{\alpha_p} : \zeta^{\alpha_1+1}) + f''.$$

We will prove the above claim. Using $\zeta^{\alpha_1+1}c - \zeta^{j+1}x_p = \zeta(\zeta^{\alpha_1}c - \zeta^j x_p)$, statement (i) follows from a straightforward computation. We will show (ii). Let

$$\tau' = \frac{\Phi(x_p^{\alpha_p} : \zeta^{\alpha_1+1})}{(\zeta^{\alpha_1+1}c - \zeta x_p)}.$$

Then τ can be written either in the form $\tau = (\zeta^{\alpha_1+1}c - \zeta^{\alpha_p+1}x_p)\tau'$ or $\tau = x_p\tau'$. Recall that $r = p + 1$. In the former case,

$$\begin{aligned}\tau &= \zeta^{\alpha_p}\big\{(\zeta^{\alpha_1+1}c - \zeta x_p) - \zeta^{\alpha_1+1}c + \zeta^{\alpha_1-\alpha_p+1}c\big\}\tau' \\ &= \zeta^{\alpha_p}\Phi(x_p^{\alpha_p} : \zeta^{\alpha_1+1}) + \zeta^{\alpha_1+1}(1 - \zeta^{\alpha_p})c\tau'\end{aligned}$$

satisfies the desired conditions. In the latter case,

$$\tau = -\zeta^{-1}\big\{(\zeta^{\alpha_1+1}c - \zeta x_p) - \zeta^{\alpha_1+1}c\big\}\tau' = -\zeta^{-1}\Phi(x_p^{\alpha_p} : \zeta^{\alpha_1+1}) + \zeta^{\alpha_1}c\tau'$$

satisfies the desired conditions. The proof of the claim is complete.

Now, it follows from (4.0) that there exists a number $\gamma \in k \setminus \{0\}$ such that

$$\mathrm{sub}\left(\frac{\Phi(x_p^{\beta_p} : \zeta^{\alpha_1})}{\tau}; x_p, \zeta^{\alpha_1}c\right) = \gamma c^{\beta_p - \alpha_p}.$$

Then, by Claim (i) and Lemma 4.5.5, the polynomial $\Phi(x_1^{\alpha_1})\{\Phi(x_p^{\beta_p} : \zeta^{\alpha_1}) - \gamma\tau c^{\beta_p-\alpha_p}\}$ is divisible by $\Phi(x_1^{\alpha_1})\tau(x_p - \zeta^{\alpha_1}c) = -\zeta^{-1}\Phi(x_1^{\alpha_1}x_p^{\alpha_p+1} : \zeta^{\alpha_1+1})$. By Lemma 4.8.2(i), we obtain

$$\Phi(x_1^{\alpha_1})x_{p+1}^{\beta_{p+1}}c^{q-\alpha_1-\beta_p-\beta_{p+1}}\big\{\Phi(x_p^{\beta_p} : \zeta^{\alpha_1}) - \gamma\tau c^{\beta_p-\alpha_p}\big\} \in \mathcal{V}(G).$$

Hence, by (4.8),

$$\gamma\,\Phi(x_1^{\alpha_1})x_{p+1}^{\beta_{p+1}}c^{q-\alpha_1-\alpha_p-\beta_{p+1}}\tau + f' \in \mathcal{V}(G).$$

Furthermore, by Claim (ii), there exists a f'' with $\mathrm{in}_{\prec_{lex}}(f'') = x_p^{\alpha_p-1}x_{p+1}$ and such that

$$(4.9)\quad \gamma\,\Phi(x_1^{\alpha_1}x_p^{\alpha_p} : \zeta^{\alpha_1+1})x_{p+1}^{\beta_{p+1}}c^{q-\alpha_1-\alpha_p-\beta_{p+1}} + f' + \gamma\,\Phi(x_1^{\alpha_1})x_{p+1}^{\beta_p+1}c^{q-\alpha_1-\alpha_p-\beta_{p+1}}f''$$

is contained in $\mathcal{V}(G)$. Since $r = p+1$ and

$$\mathrm{in}_{\prec_{lex}}\big(\Phi(x_1^{\alpha_1})x_{p+1}^{\beta_{p+1}}c^{q-\alpha_1-\alpha_p-\beta_{p+1}}f''\big) = x_1^{\alpha_1}x_p^{\alpha_p-1}x_{p+1}^{q-\alpha_1-\alpha_p+1} \prec_{lex} \tilde{g},$$

the polynomial (4.9) satisfies the desired conditions. $\square$

Lemma 4.8.5. *For every monomial $w \in k[x_r, \ldots, x_n]$ of degree $q - \alpha_1 - \alpha_{r-1}$ with $x_1^{\alpha_1}x_{r-1}^{\alpha_{r-1}}w \neq \tilde{g}$, we have that*

$$\Phi(x_1^{\alpha_1}x_{r-1}^{\alpha_{r-1}} : \zeta^{\alpha_1+1})w \in \mathcal{V}(G).$$

Proof: If $x_1^{\alpha_1}x_{r-1}^{\alpha_{r-1}}w$ is not a P-free monomial, then some $x_t^{a_t}$ with $t \geq r$ divides w. Then $\Phi(x_1^{\alpha_1}x_{r-1}^{\alpha_{r-1}} : \zeta^{\alpha_1+1})w \in \mathcal{V}(G)$ is clear since $x_t^{a_t} \in \Phi(P)$ if $t \geq r$. Suppose that $x_1^{\alpha_1}x_{r-1}^{\alpha_{r-1}}w$ is a P-free monomial. Then $x_1^{\alpha_1}x_{r-1}^{\alpha_{r-1}}w \in \tilde{L}$ since it is lex-greater than $\tilde{g}$. Thus, by Lemma 4.5.7(ii), we have

$$\Phi(x_1^{\alpha_1}x_{r-1}^{\alpha_{r-1}}w : \zeta^{\alpha_1+1}) = \Phi(x_1^{\alpha_1}x_{r-1}^{\alpha_{r-1}} : \zeta^{\alpha_1+1})w \in \mathcal{V}(\tilde{L}) \subset \mathcal{V}(G),$$

as desired. $\square$

Now, we are in the position to prove Lemma 4.6.4. Recall that $c = x_r + \cdots + x_n$. By Lemma 4.8.4, there exists a polynomial h such that $\mathrm{in}_{\prec_{lex}}(h) \prec_{lex} \tilde{g}$ and

$$\Phi(x_1^{\alpha_1}x_{r-1}^{\alpha_{r-1}} : \zeta^{\alpha_1+1})x_r^{\beta_r}c^{q-\alpha_1-\alpha_{r-1}-\beta_r} + h \in \mathcal{V}(G).$$

It follows from Lemma 4.8.1(ii) and the definition of r that the monomial $x_1^{\alpha_1} x_{r-1}^{\alpha_{r-1}} x_r^{\beta_r}$ divides $\tilde{g}$ and $\dfrac{\tilde{g}}{x_1^{\alpha_1} x_{r-1}^{\alpha_{r-1}}} \in k[x_r, \ldots, x_n]$. This fact implies that $x_r^{\beta_r} c^{q-\alpha_1-\alpha_{r-1}-\beta_r}$ can be written in the form

$$x_r^{\beta_r} c^{q-\alpha_1-\alpha_{r-1}-\beta_r} = x_r^{\beta_r}(x_r + \cdots + x_n)^{q-\alpha_1-\alpha_{r-1}-\beta_r} = \delta\left(\frac{\tilde{g}}{x_1^{\alpha_1} x_{r-1}^{\alpha_{r-1}}}\right) + \tilde{h},$$

where $\delta \in k \setminus \{0\}$ and where $\tilde{h}$ is a k-linear combination of monomials of $k[x_r, \ldots, x_n]$ which is not $\dfrac{\tilde{g}}{x_1^{\alpha_1} x_{r-1}^{\alpha_{r-1}}}$. Furthermore, since Lemma 4.8.5 implies $\Phi(x_1^{\alpha_1} x_{r-1}^{\alpha_{r-1}} : \zeta^{\alpha_1+1})\tilde{h} \in \mathcal{V}(G)$, it follows that

$$\Phi(x_1^{\alpha_1} x_{r-1}^{\alpha_{r-1}} : \zeta^{\alpha_1-1})\left(\frac{\tilde{g}}{x_1^{\alpha_1} x_{r-1}^{\alpha_{r-1}}}\right) + h \in \mathcal{V}(G).$$

The initial monomial of the above polynomial is $\tilde{g}$. $\square$

4.9. Proof of Theorem 1.5. First, we recall the definition of consecutive cancellation, which we will use. Given a sequence of numbers $\{c_{i,j}\}$, we obtain a new sequence by a *cancellation* as follows: fix a j, and choose i and i' so that one of the numbers is odd and the other is even; then replace $c_{i,j}$ by $c_{i,j} - 1$, and replace $c_{i',j}$ by $c_{i',j} - 1$. We have a *consecutive cancellation* when $i' = i + 1$. The term "consecutive" is justified by the fact that we consider cancellations in Betti numbers of consecutive homological degrees. The following result is proved in [Pe]: if Q is a graded ideal in S and L is the lex ideal with the same Hilbert function, then the graded Betti numbers $\beta_{i,j}^S(S/Q)$ can be obtained from the graded Betti numbers $\beta_{i,j}^S(S/L)$ by a sequence of consecutive cancellations.

In order to prove Theorem 1.5 we need the following lemmas.

Lemma 4.9.1. *Let I be a monomial+P ideal in S and $\mathcal{A} \subset \{x_1, \ldots, x_n\}$. Let $I' = \operatorname{pol}_{\mathcal{A}}(I)$ and $P' = \operatorname{pol}_{\mathcal{A}}(P)$. We have equalities of Betti numbers*

$$\beta_{ij}^{\tilde{S}/P'}(\tilde{S}/I') = \beta_{ij}^{S/P}(S/I) \qquad \textit{for all } i, j \geq 0\,.$$

Lemma 4.9.2. [GHoP, Proposition 2.6] *Let A be a homogeneous ideal in S, and let $B \supseteq A$ be another homogeneous ideal in S. Let $\prec$ be a monomial order in S. The* *graded Betti numbers of $S/\mathrm{in}_\prec(B)$ over the quotient ring $S/\mathrm{in}_\prec(A)$ are greater than or equal to those of S/B over the ring S/A. Furthermore, the graded Betti numbers of S/B can be obtained from those of $S/\mathrm{in}_\prec(B)$ by a sequence of consecutive cancellations.*

Applying the above two lemmas, we obtain the following result.

Lemma 4.9.3. *Let I and J be monomial+P ideals of S. Suppose that there exist an $\mathcal{A} \subset \{x_1, \ldots, x_n\}$, a linear transformation ϕ over $\tilde{S}$ and a monomial order $\prec$ on $\tilde{S}$ such that* $\mathrm{mingens}(J) = \mathrm{mingens}(\mathrm{in}_\prec(\phi(\mathrm{pol}_{\mathcal{A}}(I))))$ *and* $\mathrm{in}_\prec(\phi(\mathrm{pol}_{\mathcal{A}}(P))) = P\tilde{S}$*. Then*

$$\beta_{ij}^{S/P}(S/I) \leq \beta_{ij}^{S/P}(S/J) \qquad \text{for all } i, j \geq 0\,.$$

Furthermore, the Betti numbers $\beta_{ij}^{S/P}(S/I)$ can be obtained from the Betti numbers $\beta_{ij}^{S/P}(S/J)$ by a sequence of consecutive cancellations.

Proof: By Lemma 4.9.1, we get

$$\beta_{ij}^{S/P}(S/I) = \beta_{ij}^{\tilde{S}/\mathrm{pol}_{\mathcal{A}}(P)}\big(\tilde{S}/\mathrm{pol}_{\mathcal{A}}(I)\big) = \beta_{ij}^{\tilde{S}/\phi(\mathrm{pol}_{\mathcal{A}}(P))}\big(\tilde{S}/\phi\big(\mathrm{pol}_{\mathcal{A}}(I)\big)\big)$$

for all $i, j \geq 0$. Then we apply Lemma 4.9.2 and get

$$\begin{aligned}\beta_{ij}^{\tilde{S}/\phi(\mathrm{pol}_{\mathcal{A}}(P))}\big(\tilde{S}/\phi\big(\mathrm{pol}_{\mathcal{A}}(I)\big)\big) &\leq \beta_{ij}^{\tilde{S}/\mathrm{in}_\prec(\phi(\mathrm{pol}_{\mathcal{A}}(P)))}\big(\tilde{S}/\mathrm{in}_\prec\big(\phi\big(\mathrm{pol}_{\mathcal{A}}(I)\big)\big)\big)\\ &= \beta_{ij}^{\tilde{S}/(P\tilde{S})}\big(\tilde{S}/(J\tilde{S})\big)\\ &= \beta_{ij}^{S/P}(S/J)\end{aligned}$$

for all $i, j \geq 0$. Also, the second statement follows from Lemma 4.9.2 since the inequality only appears in the first line of the above computation. $\square$

Lemma 4.9.4. *Let I be a monomial+P ideal of S which is not lex+P. Then there exists a monomial+P ideal J of S which has the following properties:*

(i) J has the same Hilbert function as I.

(ii) J is lex-closer than I.

(iii) $\beta_{ij}^{S/P}(S/I) \leq \beta_{ij}^{S/P}(S/J) \qquad$ *for all $i, j \geq 0$.*

(iv) The Betti numbers $\beta_{ij}^{S/P}(S/I)$ can be obtained from the Betti numbers $\beta_{ij}^{S/P}(S/J)$ by a sequence of consecutive cancellations.

Proof: If I is not Borel+P, apply Lemma 4.9.3 to the construction in [MM, Section 3]. On the other hand, if I is Borel+P then the statement follows from Proposition 4.2 and Lemma 4.9.3. $\square$

We are ready to prove Theorem 1.5 and its refinement Theorem 4.9.5.

Theorem 1.5. *Every lex ideal in W attains maximal Betti numbers among all graded ideals with the same Hilbert function.*

Theorem 4.9.5. *If V is a graded ideal in W and L is the lex ideal with the same Hilbert function, then the graded Betti numbers $\beta_{i,j}^{W}(W/V)$ can be obtained from the graded Betti numbers $\beta_{i,j}^{W}(W/L)$ by a sequence of consecutive cancellations.*

We prove the above two theorems simultaneously.

Proof: Let I be a graded ideal in S and $I \supseteq P$. Let L be the lex+P ideal having the same Hilbert function as I. It is enough to compare the Betti numbers $\beta_{ij}^{S/P}(S/I)$ and $\beta_{ij}^{S/P}(S/L)$. Clearly the initial ideal of I (with respect to any monomial order) contains P. Thus, by Lemma 4.9.2, we may assume that I is a monomial ideal.

Iteration step: If the monomial ideal I is not lex+P, by Lemma 4.9.4, there exists a monomial+P ideal J satisfying conditions (i), (ii), (iii) and (iv) of Lemma 4.9.4. Replace I by J.

Apply repeatedly the iteration step above. At each step, we obtain a monomial+P ideal which is lex-closer than the original monomial ideal. Since there exist only finitely many different monomial+P ideals with a fixed Hilbert function, it follows that the process terminates after finitely many steps. Therefore, the last ideal is lex+P. $\square$

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