Exceptional balanced triangulations on surfaces

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Abstract

Izmestiev, Klee and Novik proved that any two balanced triangulations of a closed surface F^2 can be transformed into each other by a sequence of six operations called basic cross flips. Recently Murai and Suzuki proved that among these six operations only two operations are almost sufficient in the sense that, with for finitely many exceptions, any two balanced triangulations of a closed surface F^2 can be transformed into each other by these two operations. We investigate such finitely many exceptions, called *exceptional balanced triangulations*, and obtain the list of exceptional balanced triangulations of closed surfaces with low genera. Furthermore, we discuss the subsets $\mathcal{O}$ of the six operations satisfying the property that any two balanced triangulations of the same closed surface can be connected through a sequence of operations from $\mathcal{O}$.

Keywords: balanced triangulation, closed surface, local transformation

1 Introduction

In this paper, we only consider simple graphs which have no loops and no multiple edges. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. A *triangulation* G of a closed surface F^2 is a simple graph embedded on the surface such that each face of G is bounded by a 3-cycle and any two faces share at most one edge. For a vertex v of a triangulation, the *link* of v is the boundary cycle of the union of faces meeting at v. A triangulation G of a closed surface is *balanced* if G is 3-colorable. In the remainder of the paper, $V(G)$ is sometimes decomposed into $V_R(G) \cup V_B(G) \cup V_Y(G)$, where these classes are referred as red, blue and yellow vertices of G, respectively. At that time, we will index the elements of $V_R(G)$ as r_i (or we will simply use r if a subscript is not needed); similarly we will use b_i (or b) for elements of $V_B(G)$ and y_i (or y) for elements of $V_Y(G)$. It is clear that any vertex of a balanced triangulation has even degree; the converse does not hold in general.

Izmestiev, Klee and Novik [2] proved that any two balanced triangulations of a closed surface F^2 can be transformed into each other by a sequence of six operations shown

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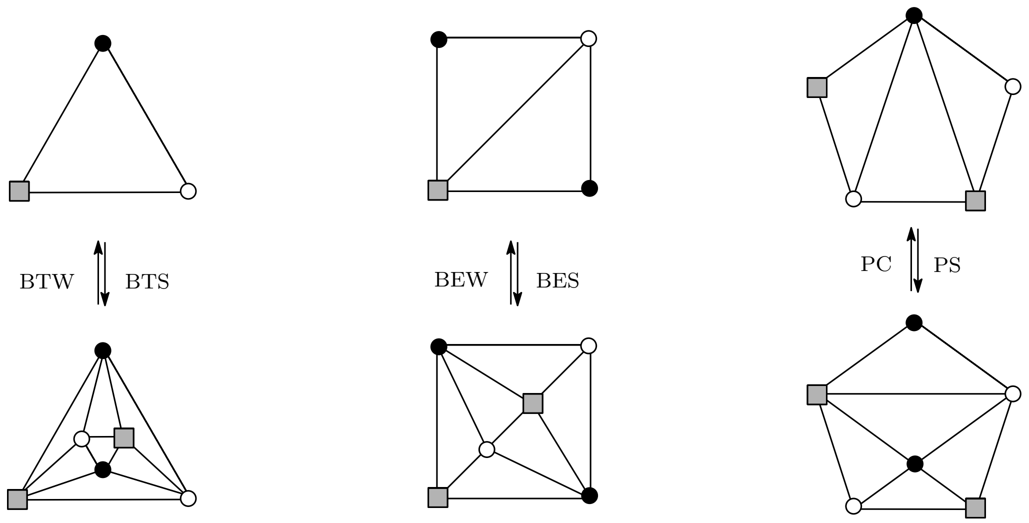

Figure 1: The six basic cross-flips.

in Figure 1, which are called *basic cross-flips* in this paper. (In this paper, we are only interested in the surface case, but we note that they actually proved that any two balanced triangulations of a closed combinatorial manifold can be connected by a sequence of cross-flips, which is an analogue of Pachner's theorem (see [7, 8]) for balanced simplicial complexes.) The six basic cross-flips are called a balanced triangle subdivision (BT-subdivision or BTS), a balanced triangle weld (BT-weld or BTW), a balanced edge subdivision (BE-subdivision or BES), a balanced edge weld (BE-weld or BEW), a pentagon splitting (P-splitting or PS) and a pentagon contraction (P-contraction or PC), respectively. A BT-subdivision (resp., -weld) and a BE-subdivision (resp., -weld) are collectively referred to as *balanced subdivisions* (resp., *-welds*). If the above operation yields multiple edges or loops, then we don't apply it since we have to preserve the simplicity of the graph. Also in [2], the authors asked if balanced subdivisions and balanced welds suffice to transform any balanced triangulation of a closed surface into any other balanced triangulation of the same surface. Murai and Suzuki [5] gave the following answer to this question.

Theorem 1 (Murai and Suzuki [5]) *For every closed surface F^2, there are balanced triangulations G and G' of F^2 such that G' cannot be obtained from G by a sequence of balanced subdivisions and welds.*

The above theorem asserts that at least one of a P-contraction and a P-splitting is necessary if a subset of the six basic cross-flips are sufficient to connect any two balanced triangulations of a given surface. Actually, the pair of a P-contraction and a P-splitting is almost sufficient to connect two given balanced triangulations on the same surface as follows.

Theorem 2 (Murai and Suzuki [5]) *For each closed surface F^2, with finitely many exceptions, any two balanced triangulations of F^2 can be transformed into each other by a sequence of P-splittings and P-contractions.*

A balanced triangulation G of a closed surface F^2 is *exceptional* if it does not admit a P-contraction or a P-splitting that preserves the simplicity of the graph. For example, it is easy to confirm that the octahedron, which is $K_{2,2,2}$ embedded on the sphere as a balanced triangulation, is exceptional. It follows from the arguments in [5] that the "finitely many exceptions" in Theorem 2 exactly correspond to the exceptional balanced triangulations of F^2. Our goal in this paper is to determine the set of exceptional balanced triangulations of several closed surfaces. The following theorem is one of our main results. Here, $\chi(F^2)$ is the Euler characteristic of the closed surface F^2.

Theorem 3 *If G is an exceptional balanced triangulation of a closed surface F^2 with $-7 \leq \chi(F^2) \leq 2$, then G is one of the following:*

(i) $K_{2,2,2}$ on the sphere (which is an octahedron).

(ii) $K_{3,3,3}$ on the torus.

(iii) $K_{4,4,4}$ on F^2 with $\chi(F^2) = -4$ (both orientable and non-orientable surfaces).

In fact, the above theorem contains some results mentioned in [5]; we shall explain those facts in our proof. By Theorem 3, we can easily obtain the following corollary for closed surfaces admitting no exceptional balanced triangulations.

Corollary 4 *Any two balanced triangulations of a closed surface which is one of the projective plane, the Klein bottle, a closed surface F^2 with $\chi(F^2) \in \{-1, -2, -3, -5, -6, -7\}$ can be transformed into each other by a sequence of P-splittings and P-contractions.*

The paper is organized as follows. In the next section, we prepare some propositions and lemmas at first, and eventually prove Theorem 3. In Section 3, we discuss the subset of the six basic cross-flips satisfying our desired property.

2 Proof of Theorem 3

By the definition of exceptional balanced triangulations, the following proposition is easily obtained, which was also put in [5] as Proposition 4.4.

Proposition 5 *A balanced triangulation G is exceptional if and only if, for any collection of three faces sharing a vertex, vwx, vxy, and vyz, the edge wz is also an edge of G.*

Because the vertices w and z in the statement of Proposition 5 have opposite colors, the following result follows immediately.

Proposition 6 *A balanced triangulation of a closed surface that is realized by a complete tripartite graph is exceptional.*

First we apply Proposition 5 to show the projective plane does not admit any exceptional balanced triangulations.

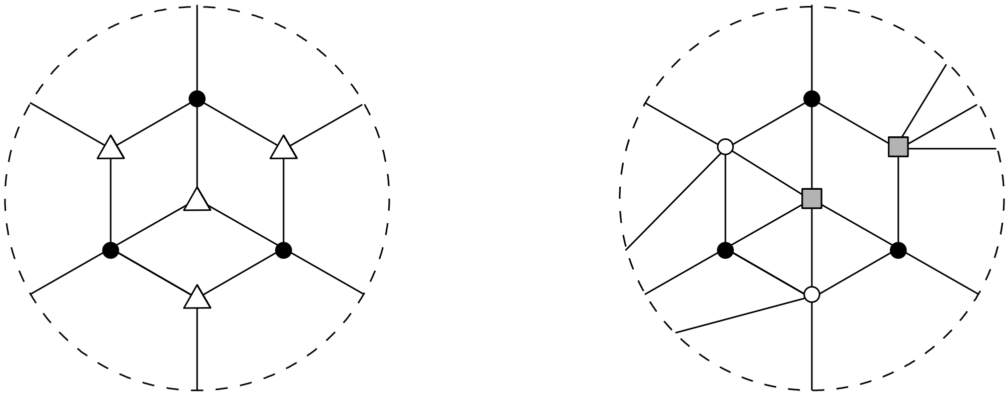

Figure 2: Configurations in the proof of Lemma 7 (1).

Lemma 7 *There exist no exceptional balanced triangulations on the projective plane.*

Proof. Assume to the contrary, that there exists an exceptional balanced triangulation G of the projective plane. By Euler's formula, G has a vertex of degree 4. Without loss of generality, we can assume it is $r \in V_R(G)$. We denote the link of r by $b_1y_1b_2y_2$; note that r is incident to four triangular faces $rb_1y_1, ry_1b_2, rb_2y_2$ and ry_2b_1. Furthermore, we assume that there are four faces $b_1y_1r_1, y_1b_2r_2, b_2y_2r_3$ and $y_2b_1r_4$ which share edges with above four faces incident to r, respectively.

Here we discuss the possibility that some of the vertices among r_1, r_2, r_3 and r_4 are identical. First, assume that two consecutive vertices, say r_1 and r_2 without loss of generality, are the same vertex. In this case, y_1 must have degree 4 because the edges r_1y_1 and r_2y_1 coincide, for otherwise, they form multiple edges. Moreover, applying Proposition 5 to faces b_2y_2r, b_2ry_1, and $b_2y_1r_1$, it follows that $r_1y_2 \in E(G)$. Next we claim that $r_1 = r_3$. If $r_1 \neq r_3$, then applying Proposition 5 to faces $b_2r_3y_2$, b_2y_2r, and b_2ry_1 tells us that $r_3y_1 \in E(G)$, which contradicts the fact that y_1 has degree 4. By symmetry, it must also be the case that $r_2 = r_4$, which implies $r_1 = r_2 = r_3 = r_4$ and hence G triangulates an octahedral sphere, which is a contradiction.

Therefore, we know $r_1 \neq r_2, r_2 \neq r_3, r_3 \neq r_4$ and $r_4 \neq r_1$. Now, we claim that $r_1 = r_3$ (resp., $r_2 = r_4$) and, the cycle $r_1b_1y_2$ (resp., $r_2b_2y_2$) is non-contractible. To get a contradiction, first assume that $r_1 \neq r_3$. Since G is exceptional, we require edges r_3b_1, r_3y_1, r_1b_2 and r_1y_2 by Proposition 5. Now, the subgraph induced by vertices $r, r_1, r_3, b_1, b_2, y_1$ and y_2 is isomorphic to $K_{2,2,3}$, which has $K_{3,4}$ as its subgraph. It is known that $K_{3,4}$ admits a unique embedding into the projective plane, as shown in the left-hand side of Figure 2. (To obtain the projective plane, identify antipodal pairs of points of dashed boundary circle.) Observe that the embedding is a quadrangulation of the projective plane; i.e., each face is bounded by a cycle of length 4. Moreover, we obtain the unique embedding of $K_{2,2,3}$ into the projective plane shown in the right-hand side of Figure 2, up to symmetry, by adding edges on the above quadrangular embedding of $K_{3,4}$. However, this embedding of $K_{2,2,3}$ does not contain the configuration having six triangular faces $rb_1y_1, ry_1b_2, rb_2y_2, ry_2b_1, r_1b_1y_1$ and $r_3b_2y_2$, a contradiction.

By the above argument, we conclude that $r_1 = r_3$. Similarly, we have $r_2 = r_4$, and this implies that each of $r_1b_1y_2$ and $r_2b_1y_1$ is essential (or non-contractible); note that each of

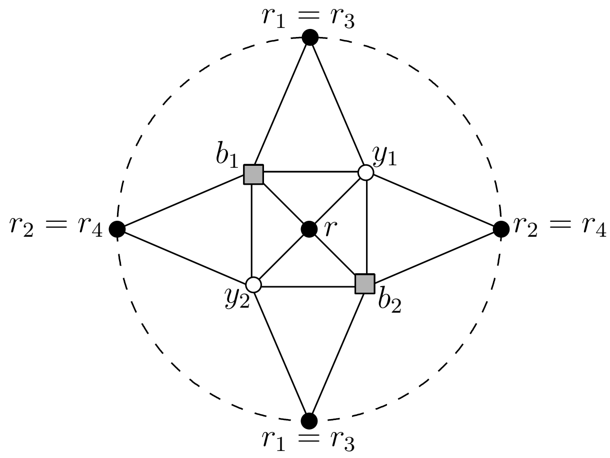

Figure 3: Configurations in the proof of Lemma 7 (2).

$r_1b_1r_2b_2$ and $r_1y_1r_2y_2$ bounds a quadrilateral 2-cell region of G (see Figure 3). Since G is balanced, the region R bounded by $r_1b_1r_2b_2$ has no diagonal edge. This implies that there exists a vertex y in R such that yr_1b_1 is a face of G, which shares the edge r_1b_1 with $y_1r_1b_1$. However, y cannot be adjacent to r, and hence G is not exceptional by Proposition 5. Therefore, the lemma follows. ■

Next, we present the following lemma which discuss exceptional balanced triangulations having no vertex of degree at least 8.

Lemma 8 *Let G be an exceptional balanced triangulation of a closed surface F^2 with $\chi(F^2) \leq 0$. If $\deg(v) \leq 6$ for every $v \in V(G)$, then G is $K_{3,3,3}$ on the torus.*

Proof. Let G be an exceptional balanced triangulation of a closed surface F^2 with $\chi(F^2) \leq 0$, and assume that $\deg(v) \leq 6$ for every $v \in V(G)$. Further, let x and y be the numbers of vertices of degree 4 and 6, respectively. By Euler's formula, we can easily obtain $2|E(G)| = 6|V(G)| - 6\chi(F^2)$. Since G has only vertices of degree 4 and 6, we have $4x + 6y = 6(x+y) - 6\chi(F^2)$ and hence $2x = 6\chi(F^2)$. Since $x \geq 0$ and $\chi(F^2) \leq 0$, it must be the case that that $x = 0$ and $\chi(F^2) = 0$. This implies that G is 6-regular and F^2 is either the torus or the Klein bottle.

Now, let b_1 be a vertex of G with link $y_1r_1y_2r_2y_3r_3$. By Proposition 5, G has edges, r_1y_3, r_2y_1 and r_3y_2. Secondly, we consider the link of y_1. Since $\deg(y_1) = 6$, y_1 is adjacent to exactly three red vertices — r_1, r_2 and r_3. Thus, the link of y_1 is $b_1r_1b_2r_2b_3r_3$ where b_2 and b_3 are vertices in $V_B(G)$. Similarly, r_1b_3 and r_3b_2 are edges by Proposition 5. By the above argument, the link of r_1 is $b_1y_1b_2y_3b_3y_2$. Moreover, by repeating the same argument, we conclude that G is a balanced triangulation given by the graph in Figure 4 and F^2 is the torus. (To obtain the torus, identify two horizontal sides and vertical sides of the square in the figure, respectively.) Observe that the underlying graph of G is isomorphic to $K_{3,3,3}$. ■

On the other hand, when an exceptional balanced triangulation contains a vertex of degree at least 8, the following statement holds.

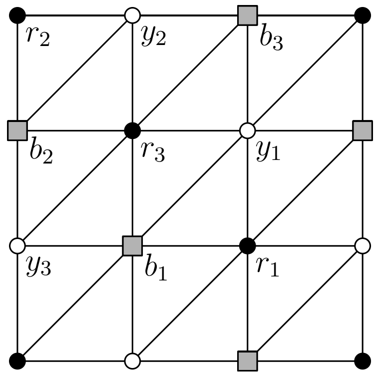

Figure 4: $K_{3,3,3}$ on the torus.

Lemma 9 *If an exceptional balanced triangulation of a closed surface F^2 has a vertex of degree at least 8, then the degree of each vertex of G is at least 8.*

Proof. Without loss of generality, we may assume $r \in V_R(G)$ is a vertex of an exceptional balanced triangulation with $\deg(r) \geq 8$. We may assume that r has link $b_1y_1b_2y_2 \cdots b_ky_k$ for $k \geq 4$. By Proposition 5, y_2b_1 and y_2b_4 are also edges in G, meaning y_2 has at least four blue neighbors and $\deg(y_2) \geq 8$ since G is balanced. The same argument applies to each neighbor of r and, since G is connected, it extends to all vertices in G. Therefore, each vertex in G has degree at least 8. ∎

The next lemma follows from Euler's formula.

Lemma 10 *If an exceptional balanced triangulation of a closed surface F^2 has a vertex of degree at least 8, then $\chi(F^2) \leq -4$. Furthermore, if $\chi(F^2) = -4$, then the underlying graph of G is isomorphic to $K_{4,4,4}$.*

Proof. Let G be an exceptional balanced triangulation of a closed surface F^2 which has a vertex of degree at least 8. Since $2|E(G)| \geq 8|V(G)|$ holds by Lemma 9, we can easily obtain $|V(G)| \leq -3\chi(F^2)$ from Euler's formula. Furthermore, each of $|V_R(G)|, |V_B(G)|$ and $|V_Y(G)|$ is at least four since $\deg(v) \geq 8$ for any $v \in V(G)$ again. That is, we have $|V(G)| \geq 12$ by the above argument. By combining these inequalities, we obtain $\chi(F^2) \leq -4$. If $\chi(F^2) = -4$, that is, all the equalities holds in the above inequalities, then we have $|V(G)| = 12$, and hence $|V_R(G)| = |V_B(G)| = |V_Y(G)| = 4$. Since the degree of each vertex is exactly 8, the underlying graph of G is isomorphic to $K_{4,4,4}$. ∎

It follows from the results in [1, 9] that $K_{4,4,4}$ can be embedded on both closed orientable surfaces of genus 3 and the closed nonorientable surface of genus 6. The following lemma is the last one before proving our main result.

Lemma 11 *There exist no exceptional balanced triangulations on a closed surface F^2 with $\chi(F^2) \in \{-5, -6, -7\}$.*

Proof. Assume to the contrary that G is an exceptional balanced triangulation on a closed surface F^2 with $\chi(F^2) \in \{-5, -6, -7\}$. Assume that the vertex set of G can be decomposed into $V(G) = V_R(G) \cup V_B(G) \cup V_Y(G)$ where $V_R(G) = \{r_1, \dots, r_i\}, V_B(G) = \{b_1, \dots, b_j\}$ and $V_Y(G) = \{y_1, \dots, y_k\}$. By Lemmas 8 and 9, each of i, j and k is at least 4. By Euler's formula, we obtain the equation $\sum_{\ell \geq 1} \ell \cdot V_{2l+6} = -3\chi(F^2)$, where V_d is the number of vertices of degree d.

We first prove that one of i, j and k is exactly 4. Suppose to the contrary that $i, j, k \geq 5$. Let V_d^R (resp., V_d^B, V_d^Y) denote the number of red (resp., blue, yellow) vertices having degree d. Further, we put $\sigma^r = \sum_{\ell \geq 1} \ell \cdot V^R_{2\ell+6}$, $\sigma^b = \sum_{\ell \geq 1} \ell \cdot V^B_{2\ell+6}$ and $\sigma^y = \sum_{\ell \geq 1} \ell \cdot V^Y_{2\ell+6}$. Since $\sum_{\ell \geq 1} \ell \cdot V_{2\ell+6} = -3\chi(F^2)$, we have

$$\sigma^r + \sigma^b + \sigma^y = -3\chi(F^2) \leq 21. \tag{1}$$

Since G has at least 15 vertices by assumption, the number of vertices having degree at least 10 is at most 6. Hence we may assume that there exists a red vertex of degree 8 and a blue vertex of degree 8. We may assume that $r_1 \in V_R(G)$ has degree 8 and the link of r_1 is $C_1 = b_1y_1b_2y_2b_3y_3b_4y_4$. By Proposition 5, any $y \in \{y_1, \dots, y_4\}$ is adjacent to b_1, b_2, b_3, b_4, so if y is adjacent to $b_5 \notin V(C_1)$ then it has degree at least 10. Since b_5 is adjacent to at least 4 yellow vertices, we have

$$(|V_Y(G)| - 4) + |\{y \in \{y_1, y_2, y_3, y_4\} : \deg(y) \geq 10\}| \geq 4.$$

Thus we obtain

$$\sigma^y \geq |V_Y(G)| + \sum_{\ell \geq 2} V^Y_{2\ell+6} \geq |V_Y(G)| + |\{y \in \{y_1, y_2, y_3, y_4\} : \deg(y) \geq 10\}| \geq 8.$$

By switching the role of yellow and blue colors, the same argument shows $\sigma^b \geq 8$. Moreover, by considering the link of a blue vertex of degree 8, we get $\sigma^r \geq 8$. This implies

$$-3\chi(F^2) = \sigma^r + \sigma^b + \sigma^y \geq 24,$$

contradicting Eq. (1). Hence one of i, j and k must be 4.

Suppose $i = 4$, that is, there are only 4 red vertices. Since each vertex has degree at least 8, each blue and yellow vertex is adjacent to all the red vertices and must have degree 8 as well. This in particular tells that each red vertex is adjacent to all the blue and yellow vertices. Hence $j = k$ and $\deg(r) = 2j$ for any vertex $r \in V_R(G)$. Combining all these facts, we conclude

$$\sigma^r + \sigma^b + \sigma^y = (j - 3) \cdot 4 + j + k = 6j - 12.$$

Hence by Eq. (1) we have $6j - 12 = -3\chi(F^2)$. Since we assumed $-7 \leq \chi(F^2) \leq -5$, we conclude that $\chi(F^2) = -6$ and $j = k = 5$.

Let $E_{R,B}(G)$ denote the set of edges connecting a red vertex to a blue vertex and define $E_{R,Y}(G)$ and $E_{B,Y}(G)$ analogously. Note that if we remove all yellow vertices from G, we obtain an even embedding of F^2 with $|V_R(G)| + |V_B(G)|$ vertices, $|E_{R,B}(G)|$ edges, and $|V_Y(G)|$ faces so that $|V_R(G)| + |V_B(G)| - |E_{R,B}(G)| + |V_Y(G)| = \chi(F^2)$, or equivalently,

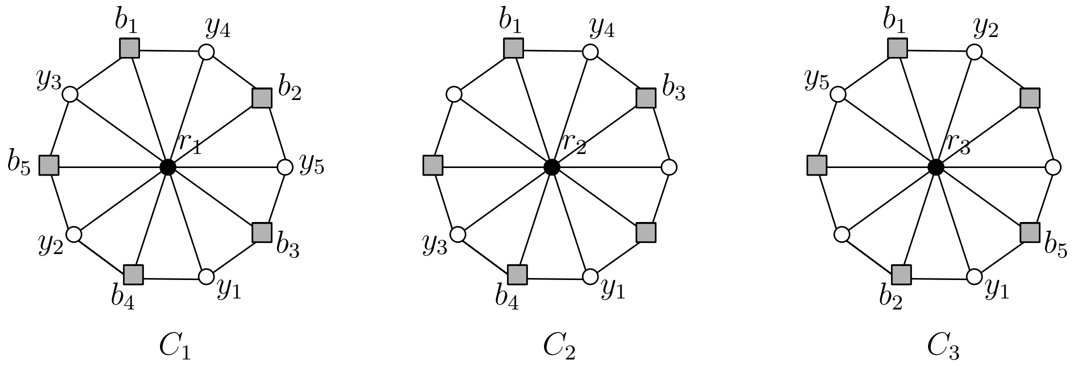

Figure 5: C_1, C_2 and C_3 in Case A

$|E_{R,B}(G)| = |V(G)| - \chi(F^2)$. Applying the same argument to $E_{R,Y}(G)$ and $E_{B,Y}(G)$, we see that

$$|E_{R,B}(G)| = |E_{R,Y}(G)| = |E_{B,Y}(G)| = |V(G)| - \chi(F^2) = 20.$$

Since $|V_R(G)| = 4$ and $|V_B(G)| = |V_Y(G)| = 5$, it must be the case that each red vertex is adjacent to every blue vertex and every yellow vertex. Moreover, since each blue (respectively, yellow) vertex has exactly four red neighbors, it must also have four yellow (resp., blue) neighbors. This means that each yellow vertex has exactly one non-neighboring blue vertex and vice-versa. Without loss of generality, we can conclude that

$$b_1y_1, b_2y_2, b_3y_3, b_4y_4, b_5y_5 \text{ are the only non-edges of } G. \tag{2}$$

For a 10-cycle $C = v_0v_1 \cdots v_9$ we say that the vertices v_ℓ and $v_{\ell+5}$ (mod 10) are in *opposite position* in C. We denote by C_ℓ the link of the vertex r_ℓ. Each C_ℓ is a 10-cycle, and applying Proposition 5 to C_ℓ, it follows that vertices b_s and y_t are adjacent if and only if they are not in opposite position on C_ℓ. This fact and (2) show

$$\text{for each } s = 1, 2, 3, 4, 5, \text{ the vertices } b_s \text{ and } y_s \text{ must be in opposite position on } C_\ell. \tag{3}$$

Without loss of generality, we may assume $C_1 = b_1y_4b_2y_5b_3y_1b_4y_2b_5y_3$ (see Figure 5).

Let $r_pb_1y_4$ be the unique face sharing the edge b_1y_4 with $r_1b_1y_4$. We may assume $r_p = r_2$. Consider C_2, the link of r_2. Since $b_2y_4r_2$ cannot be a face (otherwise, the link of y_4 contains the 4-cycle $b_1r_1b_2r_2$, a contradiction), either y_4b_3 or y_4b_5 is an edge of C_2. In the argument below, we show that both cases cannot happen.

[Case A] Suppose that y_4b_3 is an edge of C_2. Then C_2 contains the segment $b_1y_4b_3$, and also $y_1b_4y_3$ in opposite side because of (3). See Figure 5. By (3), C_2 is either $b_1y_4b_3y_2b_5y_1b_4y_3b_2y_5$ or $b_1y_4b_3y_5b_2y_1b_4y_3b_5y_2$. In the latter case, the link of y_5 becomes the 4-cycle $b_2r_1b_3r_2$, which is a contradiction. Hence $C_2 = b_1y_4b_3y_2b_5y_1b_4y_3b_2y_5$. The link of b_1 is an 8-cycle and contains the segment contains $y_3r_1y_4r_2y_5$ (see Figure 5). We may assume the link of b_1 is $y_3r_1y_4r_2y_5r_3y_2r_4$ since b_1y_1 is not an edge of G. Now, consider the link C_3. Since the link of b_1 contains the segment $y_5r_3y_2$, C_3 contains the segment $y_5b_1y_2$, and also $b_5y_1b_2$ on the opposite side by (3). See Figure 5. By inspecting C_3, one can see that either b_3y_5 or b_4y_5 is an edge of C_3. The former case cannot happen since if b_3y_5 is an edge of C_3 then the link of y_5 becomes the 6-cycle $b_3r_1b_2r_2b_1r_3$ (see the shape of C_1 and C_2), a contradiction. Hence b_4y_5 is an edge of C_3. In this case, (3)

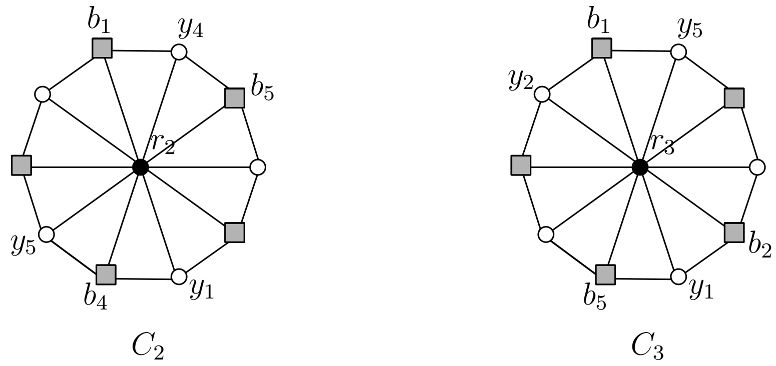

Figure 6: C_2 and C_3 in Case B

tells $C_3 = y_5b_1y_2b_3y_4b_5y_1b_2y_3b_4$. However, in this case, the link of y_3 becomes the 4-cycle $b_4r_2b_2r_3$ (see C_2 and C_3), a contradiction again.

[Case B] Suppose that y_4b_5 is an edge of C_2. Then C_2 contains the segment $b_1y_4b_5$, and $y_1b_4y_5$ on the opposite side. See Figure 6. By inspecting C_2, one can see that either y_3b_1 or y_2b_1 is an edge of C_2.

However, the former case cannot happen since if y_3b_1 is an edge of C_2 then the link of b_1 becomes the 4-cycle $y_3r_1y_4r_2$. Hence y_2b_1 is an edge of C_2, and by (3) C_2 must be $b_1y_4b_5y_3b_2y_1b_4y_5b_3y_2$. Now, observe that the link of b_1 contains the segment $y_3r_1y_4r_2y_2$. Since this link is an 8-cycle and b_1y_1 is not an edge of G, we may assume that the link of b_1 is $y_3r_1y_4r_2y_2r_3y_5r_4$. Consider the link C_3. Either b_3y_2 or b_4y_2 is an edge of C_3. See Figure 6. However, the former case cannot happen since if b_3y_2 is an edge of C_3 then the link of y_2 becomes the 4-cycle $b_1r_2b_3r_3$. Hence y_2b_4 is an edge of C_3, and by (3) the cycle C_3 must be $b_1y_5b_3y_4b_2y_1b_5y_3b_4y_2$. However, in such a case, the triangulation G contains three faces $b_3y_5r_1, b_3y_5r_2$ and $b_3y_5r_3$ sharing b_3y_5, contradicting the fact that G is a triangulation.

The above [Case A] and [Case B] complete the proof of the statement. ■

Now, all the lemmas for the proof of our main result are prepared.

Proof of Theorem 3. In [5], it was proven that the octahedron, the underlying graph of which is isomorphic to $K_{2,2,2}$, is the unique exceptional balanced triangulation of the sphere. By Lemmas 7, 8, 10 and 11, the statement of the theorem follows. ■

As we saw above, for a closed surface F^2 with $-7 \leq \chi(F^2) \leq 2$, there exist exactly three exceptional balanced triangulations, which are $K_{n,n,n}$ for $n \in \{2, 3, 4\}$ as abstract graphs. Now, one might suspect that any exceptional balanced triangulation of a closed surface is a complete tripartite graph; or more strictly $K_{n,n,n}$ with $n \geq 2$. However, the following proposition shows this is not the case.

Proposition 12 *There exists an exceptional balanced triangulation of an orientable closed surface of genus* 5*, which is not isomorphic to a complete tripartite graph.*

Proof. The graph G has five vertices of each color: $V_R(G) = \{r_1, \ldots, r_5\}, V_B(G) = \{b_1, \ldots, b_5\}$ and $V_Y(G) = \{y_1, \ldots, y_5\}$. Further, the links of red vertices are

$$r_1: \quad b_4y_4b_3y_3b_2y_5b_5y_2b_1y_1$$
$$r_2: \quad b_4y_1b_5y_3b_1y_5b_3y_4b_2y_2$$
$$r_3: \quad b_4y_2b_5y_1b_3y_5b_2y_4b_1y_3$$
$$r_4: \quad b_4y_4b_1y_1b_2y_2b_3y_3$$
$$r_5: \quad b_3y_1b_2y_3b_5y_5b_1y_2$$

Note that this is indeed an embedding since the link of every vertex is a cycle. Let E denote the set of edges of a complete tripartite graph with $V_R(G) \cup V_B(G) \cup V_Y(G)$. Then, note that $E(G) = E - \{r_4b_5, r_4y_5, r_5b_4, r_5y_4, b_4y_5, b_5y_4\}$. It is not difficult to check that G is exceptional and is embedded on an orientable closed surface of genus 5. ■

3 Other combinations of balanced transformations

In this section, we denote the set of the six basic cross-flips by $\mathcal{O}_0$; i.e., $\mathcal{O}_0 = \{\text{BES, BEW, BTS, BTW, PS, PC}\}$, and discuss the subsets $\mathcal{O} \subseteq \mathcal{O}_0$ satisfying the property that any two balanced triangulations of the same closed surface can be connected through a sequence of operations from $\mathcal{O}$. Before that, we introduce the additional operations called an *N-flip* and a P_2-flip shown in Figure 7, and the following theorem from [3], which played an important role in our argument in [5]. (Note that the N- and P_2-flips were originally defined in [6]. We can also find an N-flip in *cross-flips* used in [2].)

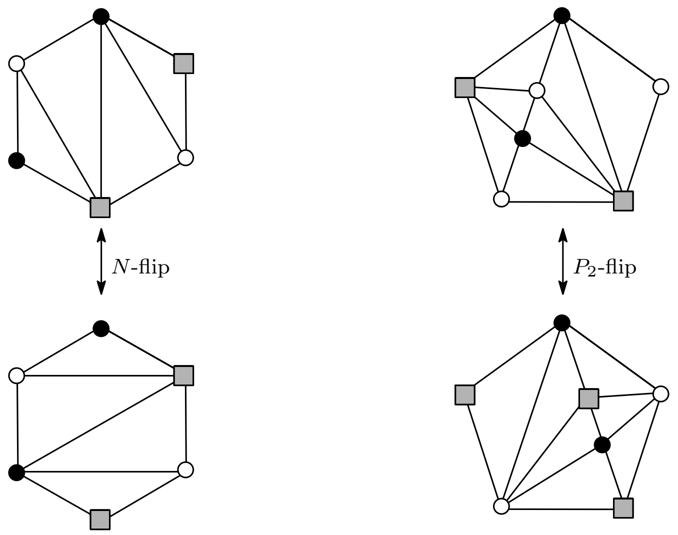

Figure 7: N-flip and P_2-flip.

Theorem 13 (Kawarabayashi, Nakamoto and Suzuki [3]) *For any closed surface F^2, there exists an integer M such that any two balanced triangulations G and G' on F^2 with $|V(G)| = |V(G')| \geq M$ can be transformed into each other by a sequence of N- and P_2-flips.*

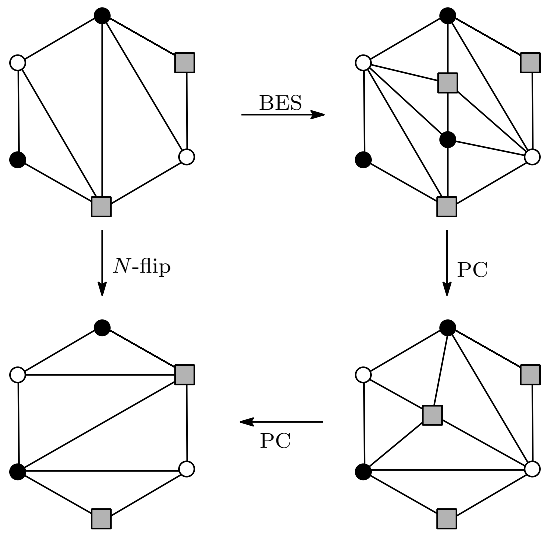

Figure 8: An N-flip realized by a sequence of a BES and PC moves.

It follows from Theorem 1 that if F^2 admits an exceptional balanced triangulation, then we need at least three operations from $\mathcal{O}_0$ to connect balanced triangulations of F^2.

Further, Murai and Suzuki [5, Theorem 1.2] showed {BES, BEW, PC} suffice to connect any two given balanced triangulations of a closed surface. Now, we give a complete answer to the question concerning the necessary subsets of $\mathcal{O}_0$.

Theorem 14 *Let F^2 be a closed surface admitting at least one exceptional balanced triangulation, and let $\mathcal{O} \subseteq \mathcal{O}_0$. Any two balanced triangulations of F^2 are transformed into each other by a sequence of elements of $\mathcal{O}$ if and only if $\mathcal{O}$ contains either $\{BES, BEW, PC\}$, $\{BES, BEW, PS\}$, $\{BTS, BTW, PC\}$, $\{BTS, BTW, PS\}$, $\{BES, BTW, PC\}$, $\{BES, BTW, PS\}$, $\{BTS, BEW, PC\}$ or $\{BTS, BEW, PS\}$.*

Proof. Since F^2 admits an exceptional balanced triangulation, $\mathcal{O}$ must contain one move from {BES, BTS} and one from {BEW, BTW}. Clearly, the above two operations are not sufficient by Theorem 1. Therefore, we need at least one from {PC, PS}. By Theorem 13 and the argument in [5], it suffices to prove that each of an N-flip and a P_2-flip can be replaced with a sequence of some operations in $\mathcal{O}$. First, see Figures 8 and 9. These figures show that {BES, PC} has the desired property. Note that all the graphs in the figure are simple. The above observation implies that an N-flip and a P_2-flip can also be replaced with a sequence of BEW and PS moves preserving the simplicity of the graph.

Furthermore, Figure 10 shows the replacement of a BES (resp. BEW) with a BTS and a PC (resp., a BTW and a PS), preserving the simplicity of the graph. This also implies that each of {BTS, PC} and {BTW, PS} also have the desired property. ■

By the observation in the previous proof, we can easily obtain the following corollary. (Note that a BES in Figures 8 and 9 can be replaced with a sequence of PS and PC moves preserving the simplicity of the graph by Proposition 5. The fact was also mentioned in [5].)

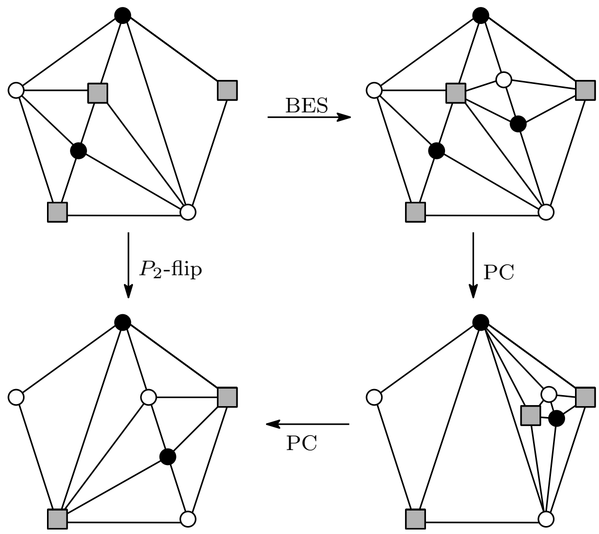

Figure 9: An P_2-flip realized by a sequence of a BES and PC moves.

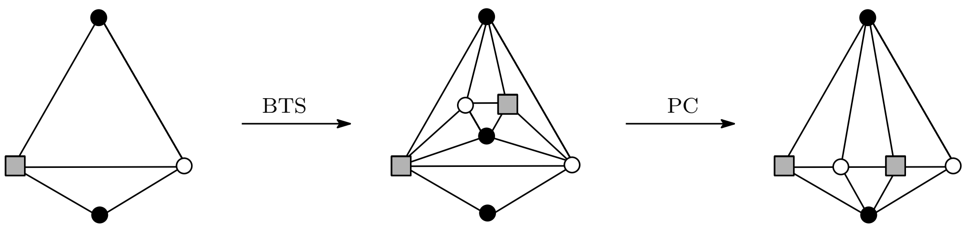

Figure 10: Replacement of a BEW with a BTW and PS.

Corollary 15 *Let F^2 be a closed surface admitting no exceptional balanced triangulation, and let $\mathcal{O} \subseteq \mathcal{O}_0$. Any two balanced triangulations of F^2 are transformed into each other by a sequence of elements of $\mathcal{O}$ if and only if $\mathcal{O}$ contains either $\{BES, PC\}$, $\{BEW, PS\}$, $\{BTS, PC\}$, $\{BTW, PS\}$ or $\{PS, PC\}$.*

For example, although {BES, PC} looks asymmetrical, it is better than {PS, PC} in the sense that any balanced triangulation on a closed surface F^2 can be transformed into another one which is not exceptional; even if F^2 admits an exceptional balanced triangulation.

Remark 16 *After the paper was accepted, we noticed that Theorem 14 was already noticed by Juhnke-Kubitzke and Venturello, and stated in [4, Example 5.3] without proofs. The theorem should be referred to as their result.*

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