BETTI TABLES OF MONOMIAL IDEALS FIXED BY PERMUTATIONS OF THE VARIABLES

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ABSTRACT. Let S_n be a polynomial ring with n variables over a field and $\{I_n\}_{n\geq 1}$ a chain of ideals such that each I_n is a monomial ideal of S_n fixed by permutations of the variables. In this paper, we present a way to determine all nonzero positions of Betti tables of I_n for all large intergers n from the $\mathbb{Z}^m$-graded Betti tables of I_m for some small integers m. Our main result shows that the projective dimension and the regularity of I_n eventually become linear functions on n, confirming a special case of conjectures posed by Le, Nagel, Nguyen and Römer.

1. INTRODUCTION

Recently, ideals fixed by an action of the infinite symmetric group in a polynomial ring with infinitely many variables attract the interest of researchers in various areas of mathematics. For example, a finite generation property of such ideals up to symmetry has been interested in algebraic statistics (see [AH, HS] and a survey [Dr]). From representation theory point of view, a study of such ideals can be considered as a special instance of twisted commutative algebra [SS1, SS2] and FI-modules [CEF]. In this paper, motivated by commutative algebra questions posed by Le, Nagel, Nguyen and Römer [LNNR1, LNNR2], we study Betti tables of monomial ideals fixed by permutations of the variables.

Let $S_\infty = \mathbb{k}[x_j : j \geq 1]$ be a polynomial ring over a field $\mathbb{k}$ with infinitely many variables $x_1, x_2, \ldots$ and let $S_n = \mathbb{k}[x_1, \ldots, x_n]$. Consider the action of the infinite symmetric group $\mathfrak{S}_\infty$ to S_∞ defined by $\sigma(x_i) = x_{\sigma(i)}$ for any $\sigma \in \mathfrak{S}_\infty$, and consider an ideal $\mathcal{I} \subset S_\infty$ which is fixed by the action of $\mathfrak{S}_\infty$, that is, satisfies $\sigma(\mathcal{I}) = \mathcal{I}$ for any $\sigma \in \mathfrak{S}_\infty$. Given such an ideal $\mathcal{I}$, by setting $I_n = \mathcal{I} \cap S_n$ we obtain a chain of ideals

$$I_1 \subset I_2 \subset I_3 \subset \cdots \tag{1}$$

such that each I_n is fixed by the action of the nth symmetric group $\mathfrak{S}_n$. About such a chain, a natural interesting algebraic problem is to understand asymptotic behavior of I_n. Indeed, Le, Nagel, Nguyen and Römer [LNNR1, LNNR2] recently studied asymptotic behavior of projective dimension and regularity of I_n. They give certain linear bounds for these invariants and conjectured that they become linear functions on n for $n \gg 0$. The above setting of considering $\mathfrak{S}_\infty$-invariant ideals in S_∞ is actually a special case of their problems since they actually discussed chains $\{I_n\}$ of ideals such that each I_n is an ideal of $\mathbb{k}[x_{i,j} : 1 \leq i \leq c, 1 \leq j \leq n]$. But even for chains arising from ideals in S_∞ asymptotic behavior of these homological invariants are still not understood very well. The purpose of this paper is to explain that, in the special case when $\mathcal{I}$ is a monomial ideal in S_∞, the situation becomes quite simple and one can describe asymptotic behavior of not only projective dimension and regularity but also the shape of the Betti table.

To explain our main result, let us first introduce a few notation and give one simple example. A **partition of length** k is a sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ of positive integers satisfying $\lambda_1 \geq \cdots \geq \lambda_k$. For a vector $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{Z}_{\geq 0}^k$, we write $x^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$. We say that an ideal $\mathcal{I} \subset S_\infty$ is **symmetric** if it is fixed by the action of $\mathfrak{S}_\infty$. We can consider that a symmetric monomial ideal $\mathcal{I} \subset S_\infty$ is generated by a finite number of partitions in the sense that there always exist partitions $\lambda(1), \dots, \lambda(t)$ such that

$$\mathcal{I} = \big(\sigma(x^{\lambda(k)}) : k = 1, 2, \dots t,\ \sigma \in \mathfrak{S}_\infty\big)$$

(see Section 2). The (i,j)th **graded Betti number** of a homogeneous ideal $I \subset S_n$ is the number $\beta_{i,j}(I) = \dim_{\Bbbk} \operatorname{Tor}_i(I, \Bbbk)_j$. To write down graded Betti numbers, we use the **Betti table** of I, which is the table whose (i,j)th entry is the number $\beta_{i,i+j}(I)$.

Now let us give a simple example. Let $\mathcal{J} \subset S_\infty$ be the symmetric monomial ideal generated by partitions $(5,1)$ and $(2,2)$, and let $J_n = \mathcal{J} \cap S_n$ for $n \geq 1$. Thus J_n is the monomial ideal of S_n generated by the $\mathfrak{S}_n$-orbits of $x_1^5 x_2$ and $x_1^2 x_2^2$. In Figures 1 and 2 below, we list Betti tables of J_n when $n = 2, 3, 4, 5, 6$ and 9 computed by the computer algebra system Macaulay2 [GS].

J_2	0	1
4	1	-
5	-	-
6	2	2

J_3	0	1	2
4	3	-	-
5	-	2	-
6	6	9	-
7	-	-	3

J_4	0	1	2	3
4	6	-	-	-
5	-	8	-	-
6	12	24	7	-
7	-	-	12	-
8	-	-	-	4

J_5	0	1	2	3	4
4	10	-	-	-	-
5	-	20	-	-	-
6	20	50	35	5	-
7	-	-	30	4	-
8	-	-	-	20	-
9	-	-	-	-	5

J_6	0	1	2	3	4	5
4	15	-	-	-	-	-
5	-	40	-	-	-	-
6	30	90	105	30	6	-
7	-	-	60	24	-	-
8	-	-	-	60	5	-
9	-	-	-	-	30	-
10	-	-	-	-	-	6

FIGURE 1. Betti tables of J_n for $n = 2, 3, 4, 5, 6$.

J_9	0	1	2	3	4	5	6	7	8
4	36	-	-	-	-	-	-	-	-
5	-	168	-	-	-	-	-	-	-
6	72	324	882	630	504	252	72	9	-
7	-	-	252	504	-	-	-	-	-
8	-	-	-	504	420	-	-	-	-
9	-	-	-	-	630	216	-	-	-
10	-	-	-	-	-	504	63	-	-
11	-	-	-	-	-	-	252	8	-
12	-	-	-	-	-	-	-	72	-
13	-	-	-	-	-	-	-	-	9

J_9	0	1	2	3	4	5	6	7	8
4	♠								
5		♠							
6	♣	♦ ♣	♠ ♣	♣	♣	♣	♣	♣	
7			♦	♠					
8				♦	♠				
9					♦	♠			
10						♦	♠		
11							♦	♠	
12								♦	
13									♦

FIGURE 2. Betti table of J_9. Each suit represents a line segment.

Looking these tables, one can find that the shape of the nonzero positions in the Betti table of J_n looks like a union of "line segments of length $n-2$". Here, for a "line segment of length ℓ", we mean a set of the form

$$\mathcal{L}\big((i,j), c, \ell\big) = \{(i+k, j+ck) \in \mathbb{Z}^2 : k = 0, 1 \dots, \ell\}$$

for some integers $i, j, c, \ell \in \mathbb{Z}_{\geq 0}$ (one may think that, in the symbol $\mathcal{L}\big((i,j),c,\ell\big)$, the first entry (i,j) represents the starting position, c represents the slope, and ℓ represents the length). Indeed, for our example of the ideal $\mathcal{J}$, it is not hard to show (see §2.4 later)

$$\begin{aligned} &\{(i,j) \in \mathbb{Z}^2 : \beta_{i,i+j}(J_n) \neq 0\} \\ &= \mathcal{L}\big((0,4),1,n-2\big) \cup \mathcal{L}\big((0,6),0,n-2\big) \cup \mathcal{L}\big((1,6),1,n-2\big). \end{aligned} \tag{2}$$

See the right table in Figure 2. The main result of this paper is the following result which shows that this phenomenon always happens if we ignore some positions in low homological degrees.

Theorem 1.1. *Let $\mathcal{I} \subset S_\infty$ be a symmetric monomial ideal generated by partitions of length $\leq m$ and let $I_n = \mathcal{I} \cap S_n$ for $n \geq 1$. Then there is a finite set $D \subset \{0,1,\ldots,m-1\} \times \mathbb{Z}^2_{\geq 0}$ such that for any integer $n \geq m$ we have*

$$\big\{(i,j) : \beta_{i,i+j}(I_n) \neq 0\big\} = \left(\bigcup_{(i,j,c) \in D} \mathcal{L}\big((i,j),c,n-m\big) \right) \cup \big\{(i,j) : \beta_{i,i+j}(I_{m-1}) \neq 0\big\}.$$

To prove Theorem 1.1, we actually prove a $\mathbb{Z}^n$-graded version of the theorem which enable us to determine all nonzero positions of the ($\mathbb{Z}^n$-graded) Betti table of I_n from the $\mathbb{Z}^m$-graded Betti table of I_m. We do not explain this result here since the statement is not very simple. See Theorem 3.2 later.

Theorem 1.1 immediately proves the following corollary about projective dimension and regularity, which confirms a special case of conjectures given by Le, Nagel, Nguyen and Römer in [LNNR1, Conjecture 1.1] and [LNNR2, Conjecture 1.2].

Corollary 1.2. *Let $\mathcal{I} \subset S_\infty$ be a nonzero proper symmetric monomial ideal and let $I_n = \mathcal{I} \cap S_n$ for $n \geq 1$. There are integers D, W, C such that*

$$\operatorname{pd}(I_n) = n - D \text{ and } \operatorname{reg}(I_n) = Wn + C \text{ for } n \gg 0.$$

We later show that the integer W can be determined combinatorially and $\operatorname{pd}(I_n)$ stabilize in a quite early stage. See Corollary 3.7 and Proposition 3.11.

There is one more interesting consequence of Theorem 1.1. The result of Nagel and Römer in [NR, Theorem 7.7] tells that for any symmetric ideal $\mathcal{I} \subset S_\infty$ and an integer $p \geq 0$, the set

$$\{j : \beta_{p,j}(\mathcal{I} \cap S_n) \neq 0\}$$

stabilize for $n \gg 0$. Theorem 1.1 shows that the following stronger property holds when $\mathcal{I}$ is a monomial ideal.

Corollary 1.3. *Let $\mathcal{I} \subset S_\infty$ be a symmetric monomial ideal and let $I_n = \mathcal{I} \cap S_n$ for $n \geq 1$. There is an integer M such that*

$$|\{j : \beta_{p,j}(\mathcal{I} \cap S_n) \neq 0\}| = M \quad \text{for } p, n \gg 0.$$

Nagel and Römer actually proved that, for a fixed p, there are finite number of syzygies that create all pth syzygies of I_n when $n \gg 0$ (see also [Sn] for a related result). We think that the above corollary suggests a possibility that there might be a way to create not only all pth syzygies for a fixed p but also all pth syzygies for arbitrary p from a finite list of syzygies.

The results on projective dimension and regularity in this paper (Corollaries 1.2, 3.7, 3.8, 3.10 and Proposition 3.11) are also proved by Claudiu Raicu [Ra] independently by a different method. He actually find a formula of $\mathrm{pd}(I)$ and $\mathrm{reg}(I)$ for any symmetric monomial ideal I in S_n and his formula determines C, D, W in Corollary 1.2.

This paper is organized as follows: In Section 2, we discuss basic properties of symmetric monomial ideals and their $\mathbb{Z}^n$-graded Betti numbers. In Section 3, we prove our main result postponing the proof of our key combinatorial proposition, and prove some refinements of Corollary 1.2. In Section 4, we prove our key combinatorial proposition, and in Section 5 we present some open questions.

2. Preliminary

In this section, we discuss basic properties of symmetric monomial ideals and multigraded Betti numbers.

2.1. **Symmetric monomial ideals.** By the result of Aschenbrenner and Hiller [AH], for any symmetric ideal $\mathcal{I} \subset S_\infty$ there are polynomials $f_1, \dots, f_t \in S_\infty$ such that

$$\mathcal{I} = \big(\sigma(f_k) : 1 \leq k \leq t,\ \sigma \in S_\infty\big).$$

In that case, we say that $\mathcal{I}$ is generated by $f_1, \dots, f_t$. A monomial ideal of S_∞ is an ideal generated by monomials. Let Λ be the set of all partitions. For any monomial $x^{\mathbf{a}}$, there is a unique partition λ such that $x^{\mathbf{a}} = \sigma(x^\lambda)$ for some $\sigma \in \mathfrak{S}_\infty$. Thus, for any symmetric monomial ideal $\mathcal{I} \subset S_\infty$, there are partitions $\lambda(1), \dots, \lambda(t)$ such that $x^{\lambda(1)}, \dots, x^{\lambda(t)}$ generates $\mathcal{I}$. We identify each $\lambda(k)$ with the monomial $x^{\lambda(k)}$ and say that $\lambda(1), \dots, \lambda(t)$ generate $\mathcal{I}$. It is easy to see that for any symmetric monomial ideal $\mathcal{I} \subset S_\infty$, there is the unique minimal subset of Λ that generates $\mathcal{I}$. This set will be denoted by $\Lambda(\mathcal{I})$.

We also say that a monomial ideal $I \subset S_n$ is symmetric if I is fixed by the action of $\mathfrak{S}_n$. In the same way as for ideals in S_∞, for any symmetric monomial ideal $I \subset S_n$, there are partitions $\lambda(1), \dots, \lambda(t)$ of length $\leq n$ such that $I = (\sigma(x^{(\lambda(k)}) : 1 \leq k \leq t, \sigma \in \mathfrak{S}_n)$. We denote by $\Lambda(I)$ the unique minimal subset of Λ that generates I. Note that if $\mathcal{I}$ is a symmetric monomial ideal of S_∞, then the ideal $I_n = \mathcal{I} \cap S_n$ is a symmetric monomial ideal of S_n with

$$\Lambda(I_n) = \{\lambda \in \Lambda(\mathcal{I}) : \lambda \text{ has length } \leq n\}.$$

For example, if $\mathcal{I}$ is generated by $(3,3)$ and $(2,2,2)$, then

$$I_1 = \{0\},\ I_2 = (x_1^3x_2^3),\ I_3 = (x_1^3x_2^3, x_1^3x_3^3, x_2^3x_3^3, x_1^2x_2^2x_3^2), \dots.$$

We often use the following property of symmetric monomial ideals.

Lemma 2.1. *Let $I \subset S_n$ be a symmetric monomial ideal generated by partitions of length $\leq m$. Suppose $n > m$. For any vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$ with $a_1, \dots, a_m \geq a_n$, if $x^{\mathbf{a}} \in I$ then $x_1^{a_1} \cdots x_{n-1}^{a_{n-1}} \in I$.*

Proof. Since I is generated by partitions of length $\leq m$, if $x^{\mathbf{a}} \in I$, then there is a monomial $u = x_{i_1}^{b_1} \cdots x_{i_m}^{b_m} \in I$ with $i_1 < \cdots < i_m$ that divides $x^{\mathbf{a}}$. If $i_m < n$, this monomial u clearly divides $x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}$. Suppose $i_m = n$. Then there is a variable x_k with $k \leq m$ that does not appear in $x_{i_1}^{b_1} \cdots x_{i_m}^{b_m}$ and we have $u' = x_{i_1}^{b_1} \cdots x_{i_{m-1}}^{b_{m-1}} x_k^{b_m} \in I$

by the symmetry of I. This monomial $u' \in I$ divides $x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}$ since $a_k \geq a_n \geq b_m$. □

2.2. **Betti numbers via simplicial complexes.** When studying graded Betti numbers of monomial ideals, it is standard to consider their multidegrees, that is, their $\mathbb{Z}^n$-gradings. An advantage of considering multidegrees is the fact that $\mathbb{Z}^n$-graded Betti numbers of monomial ideals can be computed from certain simplicial complexes. We quickly recall this fact.

Consider the $\mathbb{Z}^n$-grading of S_n such that the degree of x_i is the ith standard vector of $\mathbb{Z}^n$. For a finitely generated $\mathbb{Z}^n$-graded S_n-module M and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we write $M_{\mathbf{a}}$ for its graded component of degree $\mathbf{a}$ and call the numbers $\beta_{i,\mathbf{a}}(M) = \dim_{\Bbbk} \operatorname{Tor}_i(M, \Bbbk)_{\mathbf{a}}$ the **$\mathbb{Z}^n$-graded Betti numbers** of M.

A simplicial complex Δ on $[n] = \{1, 2, \dots, n\}$ is a collection of subsets of $[n]$ satisfying that $F \in \Delta$ and $G \subset F$ imply $G \in \Delta$. (We do not assume that every singleton of $[n]$ is contained in Δ.) Elements in Δ are called **faces** of Δ and faces having cardinality 1 are called **vertices** of Δ. We denote by $\widetilde{H}_i(\Delta)$ the ith reduced homology group of Δ over a field $\Bbbk$.

Let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$ and $\underline{\mathbf{a}} = (a_1 - 1, \dots, a_n - 1)$. For a monomial ideal $I \subset S_n$, we define the simplicial complex

$$\Delta^I_{\mathbf{a}} = \left\{ F \subset [n] : \frac{x^{\mathbf{a}}}{x^F} \in I \right\} = \{F \subset [n] : x^{\underline{\mathbf{a}}} \cdot x^{[n] \setminus F} \in I\}$$

where $x^F = \prod_{i \in F} x_i$. Note that we consider that $\frac{x^{\mathbf{a}}}{x^F}$ and $x^{\underline{\mathbf{a}}} \cdot x^{[n] \setminus F}$ are not in I if they have a negative exponent. The following fact is known.

Lemma 2.2. *For any monomial ideal $I \subset S_n$ and $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$, we have*

$$\operatorname{Tor}_i(I, \Bbbk)_{\mathbf{a}} \cong \widetilde{H}_{i-1}(\Delta^I_{\mathbf{a}}) \quad \textit{for all } i \geq 0.$$

Indeed, the degree $\mathbf{a}$ homogeneous component of the Koszul complex of I w.r.t. the variables $x_1, \dots, x_n$ can be identified with the simplicial chain complex of $\Delta^I_{\mathbf{a}}$. See [MS, Theorem 1.34].

Here we also recall a few basic facts on $\mathbb{Z}^n$-graded Betti numbers and homologies of simplicial complexes. Let $I \subset S_n$ be a monomial ideal and $G(I)$ the minimal set of monomial generators of I. The set of all lcms of monomials in $G(I)$ is called the **lcm lattice of** I and will be denoted by $\operatorname{Lcm}(I)$. For a vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$, we write

$$\operatorname{supp}(\mathbf{a}) = \{i : a_i > 0\}$$

and $\operatorname{supp}(x^{\mathbf{a}}) = \operatorname{supp}(\mathbf{a})$.

Lemma 2.3. *Let $I \subset S_n$ be a monomial ideal and $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$.*

(i) *If $\mathbf{a} \notin \operatorname{Lcm}(I)$ then $\beta_{i,\mathbf{a}}(I) = 0$.*
(ii) *If there is an element $m \in G(I)$ such that m divides $x^{\mathbf{a}}$ and $\operatorname{supp}(x^{\mathbf{a}}) = \operatorname{supp}(x^{\mathbf{a}}/m)$ then $\beta_{i,\mathbf{a}}(I) = 0$.*

The first statement is an easy consequence of Taylor resolutions (see [HH, §7.1]), and the second statement follows since $\Delta^I_{\mathbf{a}}$ becomes the $(n-1)$-simplex $\{F : F \subset [n]\}$ under the assumption. See the proof of [BPS, Theorem 3.2].

We also need the following fact.

Lemma 2.4. *Let $I \subset S_n$ be a monomial ideal and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$.*

(i) *If $a_n = 0$ then $\beta_{i,\mathbf{a}}(I) = \beta_{i,(a_1,\dots,a_{n-1})}(I \cap S_{n-1})$.*
(ii) *If $\beta_{i,\mathbf{a}}(I) \neq 0$ then $i < |\mathrm{supp}(\mathbf{a})|$.*

Proof. The first statement follows from the fact that $\Delta^I_{(a_1,\dots,a_{n-1},0)} = \Delta^{I \cap S_{n-1}}_{(a_1,\dots,a_{n-1})}$. We prove (ii). Let $t = |\mathrm{supp}(\mathbf{a})|$. We may assume $\mathbf{a} = (a_1, \dots, a_t, 0, \dots, 0)$. Then the first statement tells $\beta_{i,(a_1,\dots,a_t)}(I \cap S_t) = \beta_{i,\mathbf{a}}(I) \neq 0$. Since $I \cap S_t$ is an ideal in the polynomial ring with t variables, i must be smaller than $t = |\mathrm{supp}(\mathbf{a})|$. □

We say that a simplicial complex is **acyclic** if all its homology groups are zero. Let $\Delta \subset \Gamma$ be simplicial complexes. Then there is a natural map of homologies $\iota : \widetilde{H}_i(\Delta) \to \widetilde{H}_i(\Gamma)$ induced by the inclusion $\Delta \subset \Gamma$. We often use the next fact.

Lemma 2.5. *Let $\Delta \subset \Gamma \subset \Sigma$ be simplicial complexes. If Γ is acyclic, then the map $\iota : \widetilde{H}_i(\Delta) \to \widetilde{H}_i(\Sigma)$ induced by the inclusion $\Delta \subset \Sigma$ is zero for any i.*

Proof. The statement immediately follows from the fact that ι equals to the composition of the two maps $\widetilde{H}_i(\Delta) \to \widetilde{H}_i(\Gamma)$ and $\widetilde{H}_i(\Gamma) \to \widetilde{H}_i(\Sigma)$ induced by inclusions. □

2.3. **Multigraded Betti numbers of symmetric monomial ideals.** If $I \subset S_n$ is a symmetric monomial ideal, then $\mathrm{Tor}_i(I, \mathbb{k})$ admit an action of $\mathfrak{S}_n$ induced by the action on I. By this action, $\sigma \in \mathfrak{S}_n$ sends each element $f \in \mathrm{Tor}_i(I, \mathbb{k})$ of degree $(a_1, \dots, a_n)$ to an element of degree $(a_{\sigma(1)}, \dots, a_{\sigma(n)})$. Thus, to study $\mathbb{Z}^n$-graded Betti numbers of I, it is enough to consider degrees $(a_1, \dots, a_n)$ with $a_1 \geq \cdots \geq a_n$.

Let Δ be a simplicial complex. We say that Δ is a cone with apex v if

$$\Delta = \Delta' \cup \{\{v\} \cup F : F \in \Delta'\}$$

where $\Delta' = \{F \in \Delta : v \notin F\}$. It is well-known in combinatorial topology that if Δ is a cone, then Δ is acyclic. The next statement is easy to prove, but gives a strong restriction to possible multidegrees for Betti numbers of symmetric monomial ideals.

Proposition 2.6. *Let $I \subset S_n$ be a symmetric monomial ideal generated by partitions of length $\leq m$ and $\mathbf{a} = (a_1, \dots, a_t, 0, \dots, 0) \in \mathbb{Z}^n_{\geq 0}$ with $a_1 \geq \cdots \geq a_t \geq 1$. If $t > m$ and $a_t < a_m$, then $\beta_{i,\mathbf{a}}(I) = 0$ for all i.*

Proof. We prove that the simplicial complex

$$\Delta^I_{\mathbf{a}} = \{F \subset [n] : x^{\underline{\mathbf{a}}} \cdot x^{[n]\setminus F} \in I\} = \{F \subset [t] : x_1^{a_1-1} \cdots x_t^{a_t-1} x^{[t]\setminus F} \in I\}$$

is a cone with apex t. Let $F \in \Delta^I_{\mathbf{a}}$ with $t \notin F$. What we must prove is that $F \cup \{t\} \in \Delta^I_{\mathbf{a}}$. Since $F \in \Delta^I_{\mathbf{a}}$, we have

$$x_1^{a_1-1} \cdots x_t^{a_t-1} x^{[t]\setminus F} = x_1^{a_1-1} \cdots x_{t-1}^{a_{t-1}-1} \cdot x^{[t]\setminus (F \cup \{t\})} \cdot x_t^{a_t} \in I.$$

Since $m < t$ and $a_m > a_t$, Lemma 2.1 tells $(x_1^{a_1-1} \cdots x_{t-1}^{a_{t-1}-1}) x^{[t]\setminus (F \cup \{t\})} \in I$ and we have $(x_1^{a_1-1} \cdots x_{t-1}^{a_{t-1}-1} x_t^{a_t-1}) x^{[t]\setminus (F \cup \{t\})} \in I$. This tells $F \cup \{t\} \in \Delta^I_{\mathbf{a}}$ as desired. □

Another expression of Proposition 2.6 is that, with the same notation as in the proposition, if $\beta_{i,\mathbf{a}}(I_n) \neq 0$ for some i and $n \geq m$, then $\mathbf{a}$ must be of the form

$$\mathbf{a} = (a_1, \dots, a_{m-1}, a_m, a_m, \dots, a_m, 0, \dots, 0).$$

2.4. **Warm up: Ideals generated by partitions of length 2.** In this subsection, to get some feelings about (multigraded) Betti tables of symmetric monomial ideals, we discuss a very special case when the ideal is generated by partitions of length 2. This subsection can be considered as a special case of a more general result proved later and can be skipped if the reader is just interested in the proof of the main result.

For integers a and ℓ, we write $(a^\ell) = (a, \dots, a) \in \mathbb{Z}^\ell$. We also define $(a_1^{\ell_1}, \dots, a_t^{\ell_t}) \in \mathbb{Z}^{\ell_1+\cdots+\ell_t}$ similarly. Let $\mathcal{I} \subset S_\infty$ be a symmetric monomial ideal generated by partitions of length 2. Then $\Lambda(\mathcal{I})$ must be a set of the form

$$\Lambda(\mathcal{I}) = \{(p_1, q_1), (p_2, q_2), \dots, (p_t, q_t)\}$$

with

$$p_1 > p_2 > \cdots > p_t \geq q_t > \cdots > q_2 > q_1 \geq 1.$$

In this situation, all possible $\mathbb{Z}^n$-graded Betti numbers are determined in the following way.

Proposition 2.7. *Let $\mathcal{I}$ and $(p_1, q_1), \dots, (p_t, q_t)$ be as above. Fix $n \geq 2$ and let $I_n = \mathcal{I} \cap S_n$. Then $\beta_{i,\mathbf{a}}(I) \in \{0, 1\}$ for any $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$ and*

$$\begin{aligned}
&\big\{\big(i, (a_1, \dots, a_n)\big) \in \mathbb{Z} \times \mathbb{Z}^n : \beta_{i,(a_1,\dots,a_n)}(I_n) \neq 0, a_1 \geq \cdots \geq a_n\big\} \\
&= \left(\bigcup_{k=1}^{t} \big\{\big(i, (p_k, q_k^{i+1}, 0^{n-i-2})\big) : i = 0, 1, \dots, n-2\big\}\right) \\
&\quad \cup \left(\bigcup_{k=1}^{t-1} \big\{\big(i+1, (p_k, q_{k+1}^{i+1}, 0^{n-i-2})\big) : i = 0, 1, \dots, n-2\big\}\right) \cup E,
\end{aligned}$$

where $E = \big\{\big(i+1, (p_t^{i+2}, 0^{n-i-2})\big) : i = 0, 1, \dots, n-2\big\}$ if $p_t > q_t$ and $E = \emptyset$ if $p_t = q_t$.

Proof. We consider $p_{t+1} = q_t$ and $q_{t+1} = p_t$ for convenience. Let $\mathbf{a} = (a_1, \dots, a_n)$ with $a_1 \geq \cdots \geq a_n$ and suppose $\beta_{i,\mathbf{a}}(I_n) \neq 0$. Then, by Lemma 2.3(i), a_1 must equal to p_k for some k. Also, $q_2 \geq q_k$ since $x^{\mathbf{a}}$ must be divisible by some $x_1^{p_\ell} x_2^{q_\ell}$ and Lemma 2.3(ii) tells $a_2 \leq q_{k+1}$ (otherwise $x^{\mathbf{a}}$ and $x^{\mathbf{a}}/(x_1^{p_{k+1}} x_2^{q_{k+1}})$ have the same support). Hence $a_2 = q_k$ or $a_2 = q_{k+1}$, and Proposition 2.6 tells that $\mathbf{a}$ must be of the form either $\mathbf{a} = (p_k, q_k^\ell, 0^{n-\ell-1})$ or $\mathbf{a} = (p_k, q_{k+1}^\ell, 0^{n-\ell-1})$ for some integer ℓ. The desired statement follows from the following case analysis.

Case (I). Suppose $\mathbf{a} = (p_k, q_k^\ell, 0^{n-\ell-1})$ with $1 \leq k \leq t$. Then

$$\begin{aligned}
\Delta_{\mathbf{a}}^{I_n} &= \{F \subset [\ell+1] : x_1^{p_k-1} x_2^{q_k-1} \cdots x_{\ell+1}^{q_k-1} x^{[\ell+1]\setminus F} \in I_n\} \\
&= \{F \subset [\ell+1] : 1 \notin F \text{ and } ([\ell+1] \setminus F) \cap \{2, \dots, \ell+1\} \neq \emptyset\} \\
&= \{F \subset \{2, \dots, \ell+1\} : F \neq \{2, \dots, \ell+1\}\}.
\end{aligned}$$

Thus $\Delta_{\mathbf{a}}^{I_n}$ is the boundary of the $(\ell-1)$-simplex and we have $\beta_{i,\mathbf{a}}(I_n) = \dim_{\Bbbk} \widetilde{H}_{i-1}(\Delta_{\mathbf{a}}^{I_n}) \neq 0$ if and only if $i = \ell - 1$, and we also have $\widetilde{H}_{\ell-2}(\Delta_{\mathbf{a}}^{I_n}) \cong \Bbbk$.

Case (II). Suppose $\mathbf{a} = (p_k, q_{k+1}^\ell, 0^{n-\ell-1})$ with $1 \leq k \leq t-1$. Then

$$\begin{aligned}
\Delta_{\mathbf{a}}^{I_n} &= \{F \subset [\ell+1] : x_1^{p_k-1} x_2^{q_{k+1}-1} \cdots x_{\ell+1}^{q_{k+1}-1} x^{[\ell+1]\setminus F} \in I_n\} \\
&= \{F \subset [\ell+1] : 1 \notin F \text{ or } ([\ell+1] \setminus F) \cap \{2, \dots, \ell+1\} \neq \emptyset\} \\
&= \{F \subset [\ell+1] : F \neq [\ell+1]\}
\end{aligned}$$

is the boundary of the ℓ-simplex. Hence $\beta_{i,\mathbf{a}}(I_n) = \dim_{\Bbbk} \widetilde{H}_{i-1}(\Delta_{\mathbf{a}}^{I_n}) \neq 0$ if and only if $i = \ell$, and we have $\widetilde{H}_{\ell-1}(\Delta_{\mathbf{a}}^{I_n}) \cong \Bbbk$.

Case (III). Suppose $p_t > q_t$ and $\mathbf{a} = (p_t, q_{t+1}^{\ell}, 0^{n-\ell-1}) = (p_t^{\ell+1}, 0^{n-\ell-1})$. Then

$$\begin{aligned}\Delta_{\mathbf{a}}^{I_n} &= \{F \subset [\ell+1] : x_1^{p_t-1} x_2^{p_t-1} \cdots x_{\ell+1}^{p_t-1} x^{[\ell+1]\setminus F} \in I_n\} \\ &= \{F \subset [\ell+1] : F \neq [\ell+1]\}\end{aligned}$$

is the boundary of the ℓ-simplex. Hence $\beta_{i,\mathbf{a}}(I_n) = \dim_{\Bbbk} \widetilde{H}_{i-1}(\Delta_{\mathbf{a}}^{I_n}) \neq 0$ if and only if $i = \ell$, and we have $\widetilde{H}_{\ell-1}(\Delta_{\mathbf{a}}^{I_n}) \cong \Bbbk$.

□

Example 2.8. Consider the ideal $\mathcal{J} \subset S_\infty$ generated by $(5,1)$ and $(2,2)$. The previous proposition tells

$$\begin{aligned}&\big\{\big(i,(a_1,\dots,a_n)\big) : \beta_{i,(a_1,\dots,a_n)}(J_n) \neq 0, a_1 \geq \cdots \geq a_n\big\} \\ &= \big\{\big(i,(2,2^{i+1},0^{n-i-2})\big) : i = 0,1,\dots,n-2\big\} \\ &\quad \cup \big\{\big(i,(5,1^{i+1},0^{n-i-2})\big) : i = 0,1,\dots,n-2\big\} \\ &\quad \cup \big\{\big(i+1,(5,2^{i+1},0^{n-i-2})\big) : i = 0,1,\dots,n-2\big\}.\end{aligned} \tag{3}$$

The $\mathbb{Z}$-graded version of the equation (3) is nothing but the equation (2) in the introduction.

3. Main results

In Proposition 2.7 we can see that, when $\mathcal{I}$ is generated by partitions of length 2, if I_n has an ith syzygy of degree $(a_1,\dots,a_n) \in \mathbb{Z}_{\geq 0}^n$ with $a_1 \geq \cdots \geq a_n \geq 1$, then I_{n+1} has an $(i+1)$th syzygy of degree $(a_1,\dots,a_n,a_n) \in \mathbb{Z}_{\geq 0}^{n+1}$. This phenomenon actually happens to any symmetric monomial ideal.

Proposition 3.1. *Let $\mathcal{I} \subset S_\infty$ be a symmetric monomial ideal generated by partitions of length $\leq m$, and let $I_n = \mathcal{I} \cap S_n$ for $n \geq 1$. For any integer $n \geq m$ and vector $\mathbf{a} = (a_1,\dots,a_t,b,\dots,b) \in \mathbb{Z}_{\geq 0}^n$ with $a_1 \geq \cdots \geq a_t > b \geq 1$, we have*

$$\beta_{i,\mathbf{a}}(I_n) \neq 0 \ \Leftrightarrow \beta_{i+1,(\mathbf{a},b)}(I_{n+1}) \neq 0.$$

Since the proof of the above proposition is rather technical and purely combinatorial, we postpone its proof to the next section. In this section, we prove main algebraic results of this paper using Proposition 3.1.

We first prove a $\mathbb{Z}^n$-graded version of our main result. For a symmetric monomial ideal $I \subset S_n$, we define

$$\mathcal{B}(I) = \big\{\big(i,(a_1,\dots,a_n)\big) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^n : a_1 \geq \cdots \geq a_n \geq 0, \beta_{i,(a_1,\dots,a_n)}(I) \neq 0\big\}$$

and

$$\mathcal{F}(I) = \big\{\big(i,(a_1,\dots,a_n)\big) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^n : a_1 \geq \cdots \geq a_n \geq 1, \beta_{i,(a_1,\dots,a_n)}(I) \neq 0\big\}.$$

Since the action of S_n to $\mathrm{Tor}_i(I,\Bbbk)$ permutes the $\mathbb{Z}^n$-grading of its elements, the set $\mathcal{B}(I)$ determines the set $\{(i,\mathbf{a}) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^n : \beta_{i,\mathbf{a}}(I) \neq 0\}$ of nonzero positions of the $\mathbb{Z}^n$-graded Betti table of I. Also, since Lemma 2.4 tells

$$\beta_{i,(a_1,\dots,a_{n-1},0)}(I) \neq 0 \Leftrightarrow \beta_{i,(a_1,\dots,a_{n-1})}(I \cap S_{n-1}) \neq 0,$$

we have

$$\mathcal{B}(I) = \bigcup_{t=1}^{n} \big\{ \big(i,(a_1,\dots,a_t,0^{n-t})\big) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^{n} : \big(i,(a_1,\dots,a_t)\big) \in \mathcal{F}(I \cap S_t) \big\}. \tag{4}$$

Now consider a symmetric monomial ideal $\mathcal{I} \subset S_\infty$ and let $I_n = \mathcal{I} \cap S_n$ for $n \geq 1$. The equation (4) tells that, for any integer $n \geq 1$, we have

$$\mathcal{B}(I_n) = \bigcup_{t=1}^{n} \big\{ \big(i,(a_1,\dots,a_t,0^{n-t})\big) : \big(i,(a_1,\dots,a_t)\big) \in \mathcal{F}(I_t) \big\}.$$

Moreover, Propositions 2.6 and 3.1 say that if $\mathcal{I}$ is generated by partitions of length $\leq m$, then we have

$$\mathcal{F}(I_{n+1}) = \big\{ \big(i+1,(a_1,\dots,a_n,a_n)\big) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^{n+1} : \big(i,(a_1,\dots,a_n)\big) \in \mathcal{F}(I_n) \big\}$$

for $n \geq m$. In particular, $\mathcal{F}(I_m)$ determines $\mathcal{F}(I_n)$ for all $n \geq m$. These equations tell that the set $\mathcal{B}(I_n)$ is determined from the set $\mathcal{F}(I_1),\dots,\mathcal{F}(I_m)$ in the following form.

Theorem 3.2. *Let $\mathcal{I} \subset S_\infty$ be a symmetric monomial ideal generated by partitions of length $\leq m$, and let $I_n = \mathcal{I} \cap S_n$ for $n \geq 1$. Then, for any integer $n \geq m$,*

$$\begin{aligned}\mathcal{B}(I_n) = &\left(\bigcup_{t=1}^{m-1} \big\{ \big(i,(a_1,\dots,a_t,0^{n-t})\big) : \big(i,(a_1,\dots,a_t)\big) \in \mathcal{F}(I_t) \big\} \right) \\ &\cup \left(\bigcup_{k=m}^{n} \big\{ \big(i+k-m,(a_1,\dots,a_m,a_m^{k-m},0^{n-k})\big) : \big(i,(a_1,\dots,a_m)\big) \in \mathcal{F}(I_m) \big\} \right).\end{aligned}$$

Example 3.3. Consider the ideal $\mathcal{J}$ generated by partitions $(5,1)$ and $(2,2)$ in the introduction. In this case, we have

$$\begin{aligned}
\mathcal{F}(J_1) &= \emptyset,\\
\mathcal{F}(J_2) &= \big\{ \big(0,(2,2)\big), \big(0,(5,1)\big), \big(1,(5,2)\big) \big\},\\
\mathcal{F}(J_3) &= \big\{ \big(1,(2,2,2)\big), \big(1,(5,1,1)\big), \big(2,(5,2,2)\big) \big\},\\
\mathcal{F}(J_4) &= \big\{ \big(2,(2,2,2,2)\big), \big(2,(5,1,1,1)\big), \big(3,(5,2,2,2)\big) \big\},\\
&\ \ \vdots
\end{aligned}$$

Also, the set $\mathcal{B}(J_n)$ is essentially a union of $\mathcal{F}(J_1),\mathcal{F}(J_2),\dots,\mathcal{F}(J_n)$ (if we ignore zeros in degrees). For example, when $n=4$ we have

$$\mathcal{B}(J_4) = \left\{ \begin{array}{c} \big(0,(2,2,0,0)\big), \big(0,(5,1,0,0)\big), \big(1,(5,2,0,0)\big), \\ \big(1,(2,2,2,0)\big), \big(1,(5,1,1,0)\big), \big(2,(5,2,2,0)\big), \\ \big(2,(2,2,2,2)\big), \big(2,(5,1,1,1)\big), \big(3,(5,2,2,2)\big) \end{array} \right\}.$$

Now Theorem 1.1 in the introduction is an easy consequence of the above theorem.

Proof of Theorem 1.1. Theorem 3.2 says

$$\begin{aligned}
&\{(i,j) : \beta_{i,i+j}(I_n) \neq 0\} \\
&= \big\{(i, |\mathbf{a}| - i) : (i, \mathbf{a}) \in \mathcal{B}(I_n)\big\} \\
&= \bigcup_{t=1}^{m-1} \big\{(i, |\mathbf{a}| - i) : (i, \mathbf{a}) \in \mathcal{F}(I_t)\big\} \\
&\quad \cup \bigcup_{(i,(a_1,\dots,a_m))\in\mathcal{F}(I_m)} \big\{(i+k-m, a_1+\cdots+a_m + a_m(k-m)-i-k+m) : m \le k \le n\big\} \\
&= \big\{(i,j) : \beta_{i,i+j}(I_{m-1}) \neq 0\big\} \\
&\quad \cup \bigcup_{(i,(a_1,\dots,a_m))\in\mathcal{F}(I_m)} \big\{(i+k, a_1+\cdots+a_m - i + (a_m-1)k) : 0 \le k \le n-m\big\},
\end{aligned}$$

where we use $\mathcal{B}(I_{m-1}) = \bigcup_{t=1}^{m-1} \big\{\big(i, (\mathbf{a}, 0^{m-1-t})\big) : (i, \mathbf{a}) \in \mathcal{F}(I_t)\big\}$ for the last equality. By setting

$$D = \big\{(i, a_1 + \cdots + a_m - i, a_m - 1) : \big(i, (a_1, \dots, a_m)\big) \in \mathcal{F}(I_m)\big\},$$

we get the desired statement. □

Another formulation of Theorem 3.2 is

$$\begin{aligned}
\mathcal{B}(I_n) =& \big\{\big(i, (\mathbf{a}, 0^{n-m})\big) : (i, \mathbf{a}) \in \mathcal{B}(I_m) \setminus \mathcal{F}(I_m)\big\} \\
&\cup \left(\bigcup_{k=m}^{n} \big\{\big(i+k-m, (a_1, \dots, a_m, a_m^{k-m}, 0^{n-k})\big) : \big(i, (a_1, \dots, a_m)\big) \in \mathcal{F}(I_m)\big\} \right).
\end{aligned}$$

This formula enable us to determine the shape of the Betti table of I_n from $\mathbb{Z}^n$-graded Betti numbers of I_m. Below we give two examples.

Example 3.4. Let $m \geq 1$ be an integer and let $\mathcal{T} \subset S_\infty$ be the symmetric monomial ideal generated by m partitions

$$(1^m), (2^{m-1}), (3^{m-2}), \dots, (m^1).$$

Let $T_n = \mathcal{T} \cap S_n$ for $n \geq 1$. The ideal T_m is called a tree ideal, and it is known that the Scarf complex gives a minimal free resolution of T_m (see [MSY, Example 1.2 and Theorem 1.5]). This tells that the set $\{(i, \mathbf{a}) : \beta_{i,\mathbf{a}}(T_m) \neq 0\}$ equals to

$$\{(|F|-1, \mathrm{lcm}(F)) : F \subset G(T_m),\ \mathrm{lcm}(F) \neq \mathrm{lcm}(F') \text{ for any } F' \subset G(I) \text{ with } F' \neq F\}.$$

When $n > m$, a minimal free resolution of T_n cannot be given by the Scarf complex. However, one can determine the shape of its Betti table by using Theorem 3.2. For example, when $m = 4$ we have

$$\mathcal{B}(T_4) = \left\{ \begin{array}{l}
\big(0, (4,0,0,0)\big), \big(0, (3,3,0,0)\big), \big(0, (2,2,2,0)\big), \big(0, (1,1,1,1)\big), \\
\big(1, (4,3,0,0)\big), \big(1, (4,2,2,0)\big), \big(1, (4,1,1,1)\big), \\
\big(1, (3,3,2,0)\big), \big(1, (3,3,1,1)\big), \big(1, (2,2,2,1)\big), \\
\big(2, (4,3,2,0)\big), \big(2, (4,3,1,1)\big), \big(2, (4,2,2,1)\big), \big(2, (3,3,2,1)\big), \\
\big(3, (4,3,2,1)\big)
\end{array} \right\}.$$

We have 7 elements in $\mathcal{B}(T_4) \setminus \mathcal{F}(T_4)$ each of which contributes a single position in the Betti table of I_n, and we have 8 elements in $\mathcal{F}(T_4)$ each of which creates a line segment of length $n - 4$ in the Betti table. See Figure 3.

T_4	0	1	2	3
4	5	-	-	-
5	-	-	-	-
6	10	20	-	-
7	-	30	60	24

T_8	0	1	2	3	4	5	6	7
4	78	224	280	160	35	-	-	-
5	-	-	-	-	-	-	-	-
6	84	616	1400	1568	840	176	-	-
7	-	756	3976	8540	9912	6524	2296	336

T_8	0	1	2	3	4	5	6	7
4	①∗	①	①	①	①			
5								
6	∗∗	②④ ∗	②④	②④	②④	②④		
7		③∗ ∗	③⑤ ⑥⑦ ∗	③⑤ ⑥⑦ ⑧	③⑤ ⑥⑦ ⑧	③⑤ ⑥⑦ ⑧	⑤ ⑥⑦ ⑧	⑧

FIGURE 3. Betti tables of T_4 and T_8. Each numbered circle represents a line segment created by an element of $\mathcal{F}(T_4)$. The positions where elements of $\mathcal{B}(T_4) \setminus \mathcal{F}(T_4)$ contribute are marked by $*$.

P_4	0	1	2	3
10	24	36	6	-
11	-	-	8	-
12	-	-	-	-
13	-	-	-	1

P_8	0	1	2	3	4	5	6	7
10	1680	5880	7140	5040	2016	336	-	-
11	-	-	1680	1120	560	280	56	-
12	-	-	-	840	420	-	-	-
13	-	-	-	350	336	168	-	-
14	-	-	-	-	-	56	28	-
15	-	-	-	-	168	-	-	-
16	-	-	-	-	56	-	-	-
17	-	-	-	-	-	56	-	-
18	-	-	-	-	-	-	-	-
19	-	-	-	-	-	28	8	-
20	-	-	-	-	-	-	-	-
21	-	-	-	-	-	-	-	-
22	-	-	-	-	-	-	8	-
23	-	-	-	-	-	-	-	-
24	-	-	-	-	-	-	-	-
25	-	-	-	-	-	-	-	1

P_8	0	1	2	3	4	5	6	7
10	①	①② ③④	①② ③⑥	①② ③	①② ③	② ③		
11			⑤⑦ ④	⑤ ⑥	⑤	⑤	⑤	
12				④	⑥			
13				⑧⑦	④	⑥		
14						④	⑥	
15					⑦			
16					⑧			
17						⑦		
18								
19						⑧	⑦	
20								
21								
22							⑧	
23								
24								
25								⑧

FIGURE 4. Betti tables of P_4 and P_8. Numbered circles represent 8 line segments created by elements of $\mathcal{B}(P_4) = \mathcal{F}(P_4)$.

Example 3.5. The situation becomes simpler when $\mathcal{I}$ is generated by partitions of the same length m since in such a case we have $\mathcal{B}(I_m) = \mathcal{F}(I_m)$. For example, let $\mathcal{P} \subset S_\infty$ be the symmetric monomial ideal generated by a single partition $(m, m-1, \ldots, 2, 1)$ and let $P_n = \mathcal{P} \cap S_n$ for $n \geq 1$. The ideal P_m is known as a permutohedron ideal and its minimal free resolution is given by the co-Scarf complex (see [MSY, Example 4.2 and Theorem 4.6]), which tells

$$\mathcal{B}(P_m) = \big\{ \big(m-i-1, (m+1, \ldots, m+1) - \mathbf{a}^*\big) : (i, \mathbf{a}) \in \mathcal{F}(T_m) \big\},$$

where $(a_1, a_2, \ldots, a_n)^* = (a_n, \ldots, a_2, a_1)$. For example, when $m = 4$, we have

$$\mathcal{B}(\mathcal{P}_4) = \left\{ \begin{array}{l} \big(0,(4,3,2,1)\big), \big(1,(4,4,2,1)\big), \big(1,(4,3,3,1)\big), \big(1,(4,3,2,2)\big), \\ \big(2,(4,4,4,1)\big), \big(2,(4,4,2,2)\big), \big(2,(4,3,3,3)\big), \big(3,(4,4,4,4)\big) \end{array} \right\}.$$

Theorem 3.2 tells that the Betti table of P_n is a union of 8 line segments of length $n-4$ created by 8 elements in $\mathcal{F}(P_4) = \mathcal{B}(P_4)$. See Figure 4.

Finally, we discuss projective dimension and regularity of symmetric monomial ideals. Recall that, for a homogeneous ideal $I \subset S_n$, the **projective dimension** of I is the number

$$\mathrm{pd}(I) = \max\{i : \beta_{i,j}(I) \neq 0 \text{ for some } j\}$$

and the **Castelnuovo-Mumford regularity** of I is the number

$$\mathrm{reg}(I) = \max\{j : \beta_{i,i+j}(I) \neq 0 \text{ for some } i\}.$$

We first prove the following statement which is analogous to Proposition 3.1 but does not have any assumption on length of generators.

Proposition 3.6. *Let $\mathcal{I} \subset S_\infty$ be a symmetric monomial ideal and $I_n = \mathcal{I} \cap S_n$ for $n \geq 1$. Let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$ with $a_1 \geq a_2 \geq \cdots \geq a_n \geq 1$. If $\beta_{i,\mathbf{a}}(I_n) \neq 0$ then $\beta_{i+1,(\mathbf{a},k)}(I_{n+1}) \neq 0$ for some $1 \leq k \leq a_n$.*

Proof. Let

$$\Delta^{(k)} = \{F \subset [n] : x^{\mathbf{a}} \cdot x_{n+1}^k \cdot x^{[n]\setminus F} \in I_{n+1}\}$$

for $k = 0, 1, \dots, a_n$ and $\Delta = \Delta^{(0)}$. Also, let $\Sigma = \Delta \cup \{\{n\} \cup F : F \in \Delta\}$. The definition tells

(a) $\Delta = \Delta^{(0)} = \Delta^{I_n}_{\mathbf{a}}$.
(b) $\Delta^{(0)} \subset \Delta^{(1)} \subset \cdots \subset \Delta^{(a_n)}$.
(c) $\Delta^{I_{n+1}}_{(\mathbf{a},k)} = \Delta^{(k)} \cup \{\{n+1\} \cup F : F \in \Delta^{(k-1)}\}$.

Also, we have the inclusion

(d) $\Delta^{(a_n)} \supset \Sigma$.

To see (d), it is enough to show that $F \cup \{n\} \in \Delta^{(a_n)}$ for any $F \in \Delta$ with $n \notin F$. Indeed, if $F \in \Delta$ with $n \notin F$, then $x^{\mathbf{a}} \cdot x^{[n]\setminus F} \in I_n$ and we get

$$x^{\mathbf{a}} \cdot x^{[n]\setminus F} \cdot x_{n+1}^{a_n-1} = x_1^{a_1-1} \cdots x_{n-1}^{a_{n-1}-1} \cdot x^{[n-1]\setminus F} \cdot x_n^{a_n} x_{n+1}^{a_n-1} \in I_{n+1}.$$

Then by exchanging x_n and x_{n+1} using the symmetry of I_{n+1} we have

$$x^{\mathbf{a}} \cdot x^{[n]\setminus(F\cup\{n\})} \cdot x_{n+1}^{a_n} = x_1^{a_1-1} \cdots x_{n-1}^{a_{n-1}-1} \cdot x^{[n-1]\setminus F} \cdot x_n^{a_n-1} x_{n+1}^{a_n} \in I_{n+1},$$

which implies $F \cup \{n\} \in \Delta^{(a_n)}$.

Recall that we assume $\widetilde{H}_{i-1}(\Delta) \cong \mathrm{Tor}_i(I_n, \Bbbk)_{\mathbf{a}} \neq 0$. Take a nonzero element $\alpha \in \widetilde{H}_{i-1}(\Delta)$. Consider the maps

$$\widetilde{H}_{i-1}(\Delta^{(0)}) \xrightarrow{\iota_1} \widetilde{H}_{i-1}(\Delta^{(1)}) \xrightarrow{\iota_2} \cdots \xrightarrow{\iota_{a_n}} \widetilde{H}_{i-1}(\Delta^{(a_n)})$$

induced by the inclusion (b), and let $\alpha_i = \iota_i \circ \cdots \circ \iota_1(\alpha)$. Then (d) and Lemma 2.5 say that the composition $\iota_{a_n} \circ \cdots \circ \iota_1$ is a zero map, so $\alpha_{a_n} = 0$. Hence there is an integer $1 \leq k \leq a_n$ such that $\alpha_k = 0$ but α_{k-1} is nonzero, where $\alpha_0 = \alpha$. On the other hand, the equation (c) gives a long exact sequence of the pair $(\Delta^{I_{n+1}}_{(\mathbf{a},k)}, \Delta^{(k)})$ (see e.g. [Sp, §4.5])

$$\cdots \longrightarrow \widetilde{H}_i(\Delta^{I_{n+1}}_{(\mathbf{a},k)}) \xrightarrow{\eta} H_i(\Delta^{I_{n+1}}_{(\mathbf{a},k)}, \Delta^{(k)}) \cong \widetilde{H}_{i-1}(\Delta^{(k-1)}) \xrightarrow{\rho} \widetilde{H}_{i-1}(\Delta^{(k)}) \longrightarrow \cdots.$$

By a routine diagram chase computation, we can see that the image of $\alpha_{k-1} \in \widetilde{H}_{i-1}(\Delta^{(k-1)})$ by the map ρ is nothing but $\pm\alpha_k = \pm\iota_k(\alpha_{k-1})$, so α_{k-1} is contained in the kernel of ρ. Hence η is not a zero map and we get $\beta_{i+1,(\mathbf{a},k)}(I_{n+1}) = \dim_\Bbbk \widetilde{H}_i(\Delta^{I_{n+1}}_{(\mathbf{a},k)}) \neq 0$. □

Let $\mathcal{I} \subset S_\infty$ be a nonzero proper symmetric monomial ideal generated by partitions of length $\leq m$, and let $I_n = \mathcal{I} \cap S_n$ for $n \geq 1$. Theorem 1.1 says that $\mathrm{pd}(I_{n+1}) \leq \mathrm{pd}(I_n) + 1$ for $n \geq m$. On the other hand, Proposition 3.6 tells that $\beta_{i,(a_1,\dots,a_t,0,\dots,0)}(I_n) \neq 0$ with $a_1, \dots, a_t \neq 0$ implies $\beta_{i+1,(a_1,\dots,a_t,k,0,\dots,0)}(I_{n+1}) \neq 0$ for some $k \geq 1$ (use Lemma 2.4(i) when $t \neq n$). This in particular shows $\mathrm{pd}(I_{n+1}) \geq \mathrm{pd}(I_n) + 1$ when $I_n \neq \{0\}$ and $I_n \neq S_n$. Hence we get the next statement.

Corollary 3.7. *With the same notation as above,* $\mathrm{pd}(I_{n+1}) = \mathrm{pd}(I_n) + 1$ *for* $n \geq m$.

The above corollary also tells that for a symmetric monomial ideal $I \subset S_n$, the Cohen–Macaulay property of S_n/I only depends on the set $\Lambda(I)$. Recall that, for a homogeneous ideal $I \subset S_n$, the **depth** of S_n/I is the number $\mathrm{depth}(S_n/I) = n - 1 - \mathrm{pd}(I)$, and we say that S_n/I is **Cohen–Macaulay** if the Krull dimension of S_n/I is equal to its depth.

Corollary 3.8. *Let $I \subset S_n$ and $J \subset S_m$ be symmetric monomial ideals with $\Lambda(I) = \Lambda(J)$. Then S_n/I is Cohen–Macaulay if and only if S_m/J is Cohen–Macaulay.*

Proof. Corollary 3.7 tells that the depth of S_n/I only depends on $\Lambda(I)$. Thus it is enough to check that the Krull dimension of S_n/I only depends on $\Lambda(I)$. Let r be the smallest length of partitions in $\Lambda(I)$. Then the radical $\sqrt{I}$ of I is the symmetric monomial ideal generated by a single partition (1^r), in other words, $\sqrt{I}$ is the ideal generated by all squarefree monomials of degree r. Thus the Krull dimension of $S_n/\sqrt{I}$ is equal to $r-1$ (see [St, II Theorem 1.3]), which must coincide with the Krull dimension of S_n/I. Thus the Krull dimension of S_n/I only depends on $\Lambda(I)$. □

Next, we discuss Castelnuovo-Mumford regularity. The following result is a more precise version of Corollary 1.2 and solves a special case of [LNNR1, Conjecture 3.8].

Proposition 3.9. *Let $\mathcal{I} \subset S_\infty$ be a symmetric monomial ideal and let $I_n = \mathcal{I} \cap S_n$ for $n \geq 1$. Let $w = \min\{\lambda_1 : (\lambda_1, \dots, \lambda_k) \in \Lambda(\mathcal{I})\}$. There is an integer $C > 0$ such that*

$$\mathrm{reg}(I_n) = (w-1)n + C \quad \textit{for } n \gg 0.$$

Proof. Suppose that $\mathcal{I}$ is generated by partitions of length $\leq m$. Theorem 3.2 tells that there are integers W, C such that $\mathrm{reg}(I_n) = Wn + C$ for $n \gg 0$, and

$$W = \max\{a_m : \beta_{i,(a_1,\dots,a_m)}(I_m) \neq 0 \text{ for some } i \text{ and } a_1 \geq \cdots \geq a_m \geq 1\} - 1. \tag{5}$$

We will prove $W = w-1$. We first show $W \leq w-1$ (this fact was actually proved in [LNNR1, Corollary 3.5] in more general setting, but we include its proof). Since $\Lambda(I_n)$ contains a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ with $w = \lambda_1 \geq \cdots \geq \lambda_k$ and $k \leq m$, any vector $\mathbf{a} = (a_1, \dots, a_m)$ with $a_1 \geq \cdots \geq a_m \geq w+1$ satisfies $\mathrm{supp}(x^{\mathbf{a}}/x^{\lambda}) = \mathrm{supp}(x^{\mathbf{a}}/x^{\lambda})$. Thus by Lemma 2.3(ii) we have $\beta_{i,\mathbf{a}}(I_m) = 0$ for all i if $a_1 \geq \cdots \geq a_m \geq w+1$. Hence $W \leq w-1$ by (5).

We finally prove $W \geq w-1$. Let $p = \min\{k : x_1^w \cdots x_k^w x_{k+1}^{w-1} \cdots x_m^{w-1} \in I_m\}$. Then

$$\Delta^{I_m}_{(w,\dots,w)} = \{F \subset [m] : x_1^{w-1} \cdots x_m^{w-1} x^{[m]\setminus F} \in I_m\} = \{F \subset [m] : |F| \leq m-p\}.$$

This simplicial complex clearly has a nontrivial element in homological position $m-p-1$. Hence

$$\beta_{m-p,(w,\dots,w)}(I_m) = \dim_{\Bbbk} \widetilde{H}_{m-p-1}(\Delta^{I_m}_{(w,\dots,w)}) \neq 0.$$

By (5), this proves $W \geq w-1$ as desired. □

Since, in Proposition 3.11, $w=1$ is equivalent to the condition that $\mathcal{I}$ contains a squarefree monomial, we get the following simple characterization of symmetric monomial ideals whose regularity becomes constant. Note that the if part was proved in [LNNR1, Proposition 3.9] in more general setting.

Corollary 3.10. *Let $\mathcal{I} \subset S_\infty$ be a symmetric monomial ideal generated by partitions of length $\leq m$ and let $I_n = \mathcal{I} \cap S_n$ for $n \geq 1$. Then $\mathcal{I}$ contains a squarefree monomial if and only if $\operatorname{reg}(I_n)$ is constant for $n \gg 0$. Moreover, if $\operatorname{reg}(I_n)$ is constant for $n \gg 0$, then $\operatorname{reg}(I_n) = \operatorname{reg}(I_m)$ for all $n \geq m$.*

Proof. The first statement is an immediate consequence of Corollary 3.11. The second statement follows form Theorem 1.1 since if $\operatorname{reg}(I_n)$ is constant for $n \gg 0$, then any element (i,j,c) in the set D in Theorem 1.1 must satisfies $c=0$. □

The ideal $\mathcal{T}$ in Example 3.4 contains a squarefree monomial ideal so the regularity of T_n is constant for $n \geq m$.

Remark 3.11. It was pointed out by Claudiu Raicu that the number C in Proposition 3.11 can be determined as follows: Let $\mathcal{I}$, m and w be as in the proposition, and let $\alpha_n = (x_1 \cdots x_n)^{w-1}$ for $n \geq 1$. Then

$$\operatorname{reg}(I_n) = (w-1)n + \operatorname{reg}(I_m : \alpha_m) \quad \text{for } n \gg 0. \tag{6}$$

To see this, consider the short exact sequence

$$0 \longrightarrow S_n/(I_n : \alpha_n) \xrightarrow{\times \alpha_n} S_n/I_n \longrightarrow S_n/(I_n + (\alpha_n)) \longrightarrow 0.$$

Then $\operatorname{Tor}_i(S_n/(I_n+(\alpha_n)), \Bbbk)_{\mathbf{b}} = 0$ for any $\mathbf{b} \geq (w,w,\dots,w)$ (as the ideal contains $\alpha_n = x_1^{w-1}x_2^{w-1}\cdots x_n^{w-1}$ Lemma 2.3(i) implies this). Also, the regularity of I_n must be attained by some multigraded component of $\operatorname{Tor}_i(I_n,\Bbbk)$ of degree $(a_1,\dots,a_{m-1},w,\dots,w)$ for $n \gg 0$. These two facts tell $\operatorname{reg}(I_n) = (w-1)n + \operatorname{reg}(I_n : \alpha_n)$ for $n \gg 0$. Since $I_n : \alpha_n$ contains a squarefree monomial for $n \geq m$, Corollary 3.10 tells $\operatorname{reg}(I_n : \alpha_n) = \operatorname{reg}(I_m : \alpha_m)$ for $n \geq m$, which guarantees (6).

4. Proof of Proposition 3.1

In this section, we prove our key proposition Proposition 3.1. For a simplicial complex Δ and an element v (v need not to be a vertex of Δ), we define the simplicial complex $\Delta^{[v]}$ by

$$\Delta^{[v]} = \Delta \cup \{F \cup \{v\} : F \in \Delta\}.$$

This simplicial complex is always a cone with apex v, so it is an acyclic simplicial complex. See Figure 5.

For a simplicial complex Δ on $[n]$ and a subset $X \subset [n]$, we say that Δ is fixed by permutations on X if for any permutation $\sigma \in \mathfrak{S}_X$ the complex $\sigma(\Delta) = \{\sigma(F) : F \in \Delta\}$ is equal to Δ, where we consider that $\sigma(k) = k$ if $k \notin X$. For example, the simplicial complex Δ in Figure 3 is fixed by permutations on $\{1,2\}$ (and also $\{3,4\}$).

The next lemma is quite technical and its proof is not so simple, but it is crucial to prove Proposition 3.1.

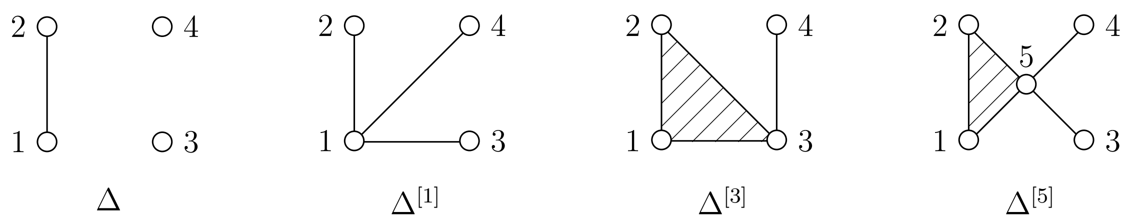

FIGURE 5. Examples of $\Delta^{[v]}$.

Lemma 4.1. *Let Δ be a simplicial complex on $[n]$, $1 \le r \le n$ an integer, and*

$$\Sigma = \Delta^{[1]} \cup \Delta^{[2]} \cup \cdots \cup \Delta^{[r]}.$$

Suppose that Δ is fixed by permutations on $[r]$. If $\widetilde{H}_i(\Sigma) \neq 0$ then $\widetilde{H}_{i-1}(\Delta) \neq 0$.

Proof. We prove the statement using induction on r. The statement is trivial when $r = 1$ since $\Delta^{[1]}$ is an acyclic simplicial complex. Suppose $r > 1$ and we assume that the statement holds for

$$\Sigma' = \Delta^{[1]} \cup \Delta^{[2]} \cup \cdots \cup \Delta^{[r-1]}.$$

Let $\Gamma = \Sigma' \cap \Delta^{[r]}$. Note that Γ contains Δ since each $\Delta^{[k]}$ contains Δ. We first prove the next claim.

Claim. $\dim_{\Bbbk} \widetilde{H}_i(\Gamma) \le \dim_{\Bbbk} H_i(\Delta)$ for all i.

Proof of Claim. Let

$$\Delta - [r] = \{F \in \Delta : F \subset \{r+1, \dots, n\}\}$$

and

$$\mathrm{st}(i) = \{F \in \Delta : \{i\} \cup F \in \Delta\}$$

for $i = 1, 2, \dots, r$. The definition tells that each $\mathrm{st}(i)$ is a cone with apex i and we have

$$\Delta = (\Delta - [r]) \cup \left(\bigcup_{k=1}^{r} \mathrm{st}(k) \right).$$

We will prove

$$\Gamma = (\Delta - [r]) \cup \left(\bigcup_{k=1}^{r} \mathrm{st}(k) \right)^{[r]}. \tag{7}$$

We first prove the inclusion "$\supset$" in (7). Recall that $\Gamma = \Sigma' \cap \Delta^{[r]}$. Since $\Delta \supset \bigcup_{k=1}^{r} \mathrm{st}(k)$, the inclusion $\Delta^{[r]} \supset (\Delta - [r]) \cup (\bigcup_{k=1}^{r} \mathrm{st}(k))^{[r]}$ is clear. It is enough to prove that any element $F \in (\bigcup_{k=1}^{r} \mathrm{st}(k))^{[r]}$ with $F \notin \Delta$ is contained in Σ'. Take such an element F. Then F must contain r since $F \notin \Delta$ and we have $F \setminus \{r\} \in \bigcup_{k=1}^{r} \mathrm{st}(k)$. Suppose $F \setminus \{r\} \in \mathrm{st}(\ell)$ with $1 \le \ell \le r$. If $\ell \notin F \setminus \{r\}$, then $(F \setminus \{r\}) \cup \{\ell\} \in \mathrm{st}_{\Delta}(\ell) \subset \Delta$, and since Δ is fixed by permutations on $[r]$ by exchanging r and ℓ in $(F \setminus \{r\}) \cup \{\ell\}$ (if $r \neq \ell$) we have $F = (F \setminus \{r\}) \cup \{r\} \in \Delta \subset \Sigma'$. On the other hand, if $\ell \in F \setminus \{r\}$, then $\ell < r$ and by exchanging ℓ and r in $F \setminus \{r\} \in \Delta$, we have $F \setminus \{\ell\} \in \Delta$ and therefore $F \in \Delta^{[\ell]} \subset \Sigma'$ as desired.

Next, we prove the inclusion '$\subset$' in (7). It is enough to prove that any element $F \in \Gamma$ with $F \notin \Delta$ is contained in $(\bigcup_{k=1}^{r} \mathrm{st}(k))^{[r]}$. Take such an $F \in \Gamma$ and suppose

$F \in \Delta^{[\ell]}$ with $\ell < r$. Since $F \in \Delta^{[\ell]} \cap \Delta^{[r]}$ and $F \notin \Delta$, F must contain both ℓ and r. Also, we have $F \setminus \{r\} \in \Delta$, and since $\ell \in F \setminus \{r\} \in \Delta$ we have $F \setminus \{r\} \in \mathrm{st}(\ell)$, which tells $F \in \mathrm{st}(\ell)^{[r]}$ as desired.

We now complete the proof of the claim. Let $X = \bigcup_{k=1}^{r} \mathrm{st}(k)$ and $Y = (\Delta - [r]) \cap X$. Any element $F \in Y$ satisfies $F \subset \{r+1, \dots, n\}$ and $F \cup \{k\} \in \Delta$ for some $1 \leq k \leq r$. But $F \cup \{k\} \in \Delta$ implies $F \cup \{1\} \in \Delta$ since Δ is fixed by permutations on $[r]$. Thus any element $F \in Y$ must satisfy $F \cup \{1\} \in \Delta$, so we have

$$Y \subset \mathrm{st}(1).$$

Also, since any element in $\Delta - [r]$ does not contain r, we have

$$(\Delta - [r]) \cap X^{[r]} = Y.$$

Compare the following two Mayer–Vietris exact sequences for $\Delta = (\Delta - [r]) \cup X$ and for $\Gamma = (\Delta - [r]) \cup X^{[r]}$,

$$\cdots \longrightarrow \widetilde{H}_i(Y) \xrightarrow{\delta_i = (\eta_i, \psi_i)} \widetilde{H}_i(\Delta - [r]) \bigoplus \widetilde{H}_i(X) \longrightarrow \widetilde{H}_i(\Delta) \xrightarrow{\rho_i} \widetilde{H}_{i-1}(Y) \xrightarrow{\delta_{i-1}} \cdots,$$

$$\cdots \longrightarrow \widetilde{H}_i(Y) \xrightarrow{\delta'_i = (\eta_i, \varphi_i)} \widetilde{H}_i(\Delta - [r]) \bigoplus \widetilde{H}_i(X^{[r]}) \longrightarrow \widetilde{H}_i(\Gamma) \xrightarrow{\rho_i} \widetilde{H}_{i-1}(Y) \xrightarrow{\delta'_{i-1}} \cdots.$$

The maps $\eta_i, \psi_i, \varphi_i$ are induced by inclusions. Since $Y \subset \mathrm{st}(1) \subset X$ and since $\mathrm{st}(1)$ and $X^{[r]}$ are acyclic, the maps ψ_i and φ_i are zero by Lemma 2.5. Hence

$$\mathrm{Image}\ \delta_i \cong \mathrm{Image}\ \delta'_i$$

for all i. Then the long exact sequences above tell

$$\begin{aligned}
\dim_{\Bbbk} \widetilde{H}_i(\Delta) &= \dim_{\Bbbk} \widetilde{H}_{i-1}(Y) + \dim_{\Bbbk} \widetilde{H}_i(\Delta - [r]) + \dim_{\Bbbk} \widetilde{H}_i(X) \\
&\quad - \dim_{\Bbbk}(\mathrm{Image}\ \delta_i) - \dim_{\Bbbk}(\mathrm{Image}\ \delta_{i-1}) \\
&\geq \dim_{\Bbbk} \widetilde{H}_{i-1}(Y) + \dim_{\Bbbk} \widetilde{H}_i(\Delta - [r]) \\
&\quad - \dim_{\Bbbk}(\mathrm{Image}\ \delta'_i) - \dim_{\Bbbk}(\mathrm{Image}\ \delta'_{i-1}) \\
&= \dim_{\Bbbk} \widetilde{H}_i(\Gamma)
\end{aligned}$$

for all i as desired. □

We now complete the proof of the Lemma 4.1. Consider the Mayer–Vietris long exact sequence for $\Sigma = \Sigma' \cup \Delta^{[r]}$

$$\cdots \longrightarrow \widetilde{H}_i(\Sigma') \bigoplus \widetilde{H}_i(\Delta^{[r]}) \longrightarrow \widetilde{H}_i(\Sigma) \longrightarrow \widetilde{H}_{i-1}(\Gamma) \longrightarrow \cdots.$$

Since $\Delta^{[r]}$ is acyclic, if $\widetilde{H}_i(\Sigma) \neq 0$, then either $\widetilde{H}_{i-1}(\Gamma) \neq 0$ or $\widetilde{H}_i(\Sigma') \neq 0$. However, the induction hypothesis tells that $\widetilde{H}_i(\Sigma') \neq 0$ implies $\widetilde{H}_{i-1}(\Delta) \neq 0$ and the claim tells that $\widetilde{H}_{i-1}(\Gamma) \neq 0$ implies $\widetilde{H}_{i-1}(\Delta) \neq 0$. Hence $\widetilde{H}_i(\Sigma) \neq 0$ implies $\widetilde{H}_{i-1}(\Delta) \neq 0$. □

We are now ready to prove Proposition 3.1. Just in case, we recall the statement.

Proposition 3.1. Let $\mathcal{I} \subset S_\infty$ be a symmetric monomial ideal generated by partitions of length $\leq m$, and let $I_n = \mathcal{I} \cap S_n$ for $n \geq 1$. For any integer $n \geq m$ and vector $\mathbf{a} = (a_1, \dots, a_t, b, \dots, b) \in \mathbb{Z}^n_{\geq 0}$ with $a_1 \geq \cdots \geq a_t > b \geq 1$, we have

$$\beta_{i,\mathbf{a}}(I_n) \neq 0 \ \Leftrightarrow \beta_{i+1,(\mathbf{a},b)}(I_{n+1}) \neq 0.$$

Proof of Proposition 3.1. We note $t < m$ by Proposition 2.6. Let

$$\Delta = \Delta_{\mathbf{a}}^{I_n} = \{F \subset [n] : x^{\mathbf{a}} \cdot x^{[n]\setminus F} \in I_n\}$$

and

$$\Sigma = \Delta_{(\mathbf{a},b)}^{I_{n+1}} = \{F \subset [n+1] : x^{\mathbf{a}} \cdot x_{n+1}^{b-1} \cdot x^{[n+1]\setminus F} \in I_{n+1}\}.$$

Clearly, Δ and Σ are fixed by permutations on $\{t+1, \dots, n\}$ and on $\{t+1, \dots, n+1\}$, respectively. We claim

$$\Sigma = \Delta^{[t+1]} \cup \Delta^{[t+2]} \cup \cdots \cup \Delta^{[n]} \cup \Delta^{[n+1]}. \tag{8}$$

To prove this equality, we first show that the righthand side is fixed by permutations on $\{t+1, \dots, n+1\}$. Let $\tau_{i,j} \in \mathfrak{S}_{n+1}$ be the transposition of i and j. Since Δ is fixed by permutations on $\{t+1, \dots, n\}$, so does $\bigcup_{k=t+1}^{n+1} \Delta^{[k]}$. Thus it is enough to show that, for any $F \in \bigcup_{k=t+1}^{n+1} \Delta^{[k]}$ and $k \in \{t+1, \dots, n\}$, we have $\tau_{k,n+1}(F) \in \bigcup_{k=t+1}^{n+1} \Delta^{[k]}$. To see this, we may assume either (i) $n+1 \in F$ and $k \notin F$, or (ii) $n+1 \notin F$ and $k \in F$. In the former case, F must be in $\Delta^{[n+1]}$ since Δ is a simplicial complex on $[n]$, and we have $F \setminus \{n+1\} \in \Delta$, which implies $\tau_{k,n+1}(F) = (F \setminus \{n+1\}) \cup \{k\} \in \Delta^{[k]}$. In the latter case, $F \in \Delta^{[\ell]}$ for some $t+1 \le \ell \le n$. If $\ell \in F$ then $F = \tau_{k,\ell}(F) \in \Delta^{[k]}$ and if $\ell \notin F$ then $F \in \Delta$. In both cases, we have $F \setminus \{k\} \in \Delta$ which implies $\tau_{k,n+1}(F) = (F \setminus \{k\}) \cup \{n+1\} \in \Delta^{[n+1]}$.

Now we prove the inclusion '$\supset$'. Let $F \in \Delta$. Since $x^{\mathbf{a}} \cdot x^{[n]\setminus F} \in I_n$, it is clear that

$$x^{\mathbf{a}} \cdot x_{n+1}^{b-1} \cdot x^{[n+1]\setminus (F \cup \{n+1\})} = x^{\mathbf{a}} \cdot x_{n+1}^{b-1} \cdot x^{[n]\setminus F} \in I_{n+1},$$

which implies $F \cup \{n+1\} \in \Sigma$. Hence $\Delta^{[n+1]} \subset \Sigma$. This also tells $\Delta^{[k]} \subset \Sigma$ for all $k = t+1, \dots, n$ since Σ is fixed by permutations on $\{t+1, \dots, n+1\}$.

Next we prove the inclusion '$\subset$' in (8). Let $F \in \Sigma$. Then

$$m = x^{\mathbf{a}} \cdot x_{n+1}^{b-1} \cdot x^{[n+1]\setminus F} \in I_{n+1}.$$

Since both simplicial complexes in (8) are fixed by permutations on $\{t+1, \dots, n+1\}$, we may assume either (i) F contains none of $t+1, \dots, n+1$, or (ii) $n+1 \in F$. The desired inclusion follows from the following case analysis.

Case (i). If F contains none of $t+1, \dots, n+1$, then

$$m = x_1^{a_1-1} \cdots x_t^{a_t-1} x_{t+1}^{b} \cdots x_{n+1}^{b} \cdot x^{[t]\setminus F} \in I_{n+1}.$$

Then Lemma 2.1 tells $x^{\mathbf{a}} \cdot x^{[n]\setminus F} = (m/x_{n+1}^b) \in I_{n+1} \cap S_n = I_n$ and we have $F \in \Delta$.

Case (ii). Suppose $n+1 \in F$. Let $F' = F \setminus \{n+1\}$. We have

$$m = x_1^{a_1-1} \cdots x_t^{a_t-1} x_{t+1}^{b-1} \cdots x_n^{b-1} \cdot x^{[n]\setminus F'} \cdot x_{n+1}^{b-1} \in I_{n+1}.$$

Again, Lemma 2.1 tells $x^{\mathbf{a}} \cdot x^{[n]\setminus F'} \in I_{n+1}$ and we have $F' \in \Delta$. Hence we get $F = F' \cup \{n+1\} \in \Delta^{[n+1]}$.

Now we prove the proposition. Let

$$\Sigma' = \Delta^{[t+1]} \cup \cdots \cup \Delta^{[n]}.$$

Then $\Sigma = \Sigma' \cup \Delta^{[n+1]}$ and $\Sigma' \cap \Delta^{[n+1]} = \Delta$ since $n+1$ is not a vertex of Σ'. Consider the Mayer–Vietris long exact sequence for $\Sigma = \Sigma' \cup \Delta^{[n+1]}$

$$\cdots \longrightarrow \widetilde{H}_i(\Delta) \xrightarrow{\delta_i = (\psi_i, \varphi_i)} \widetilde{H}_i(\Sigma') \bigoplus \widetilde{H}_i(\Delta^{[n+1]}) \longrightarrow \widetilde{H}_i(\Sigma) \longrightarrow \widetilde{H}_{i-1}(\Delta) \xrightarrow{\delta_{i-1}} \cdots.$$

Note that $\psi_i : \widetilde{H}_i(\Delta) \to \widetilde{H}_i(\Sigma')$ and $\varphi_i : \widetilde{H}_i(\Delta) \to \widetilde{H}_i(\Delta^{[n+1]})$ are the maps induced by inclusions. Since $\Delta \subset \Delta^{[1]} \subset \Sigma'$, Lemma 2.5 tells that δ_i is a zero map for all i. Then we have $\widetilde{H}_i(\Sigma) \neq 0$ if and only if $\widetilde{H}_i(\Sigma') \neq 0$ or $\widetilde{H}_{i-1}(\Delta) \neq 0$. However, Lemma 4.1 tells that $\widetilde{H}_i(\Sigma') \neq 0$ implies $\widetilde{H}_{i-1}(\Delta) \neq 0$. Hence $\widetilde{H}_i(\Sigma) \neq 0$ if and only if $\widetilde{H}_{i-1}(\Delta) \neq 0$, in particular,

$$\beta_{i,\mathbf{a}}(I_n) \neq 0 \Leftrightarrow \widetilde{H}_{i-1}(\Delta) \neq 0 \Leftrightarrow \widetilde{H}_i(\Sigma) \neq 0 \Leftrightarrow \beta_{i+1,(\mathbf{a},b)}(I_{n+1}) \neq 0,$$

as desired. □

5. Questions and Problems

In this section we pose some questions.

Algebraic proof. Our proof of Theorem 1.1 is based on careful analysis of homologies of simplicial complexes, and we do not see any *algebraic* reason that explains why we get such a simple stability for the shape of Betti tables. Thus we ask the following question.

Question 5.1. Is there a purely algebraic proof of Theorem 1.1?

Betti numbers. Let $\mathcal{I} \subset S_\infty$ be a symmetric monomial ideal generated by partitions of length $\leq m$ and $I_n = \mathcal{I} \cap S_n$ for $n \geq 1$. We determine the shape of the (multigraded) Betti table of I_n, however we could tell nothing about the values of Betti numbers.

Question 5.2. With the same notation as above, is there a way to determine the numbers $\beta_{i,j}(I_n)$ for all i, j and $n \geq m$?

To study such a problem, it might be important to understand the $\Bbbk[\mathfrak{S}_n]$-module structure of $\mathrm{Tor}_i(I_n, \Bbbk)_j$ rather than just determining its $\Bbbk$-dimension. Such a structure is studied for some particular monomial ideals in [Ga, BdAG+]. Also, a more challenging problem would be

Problem 5.3. Find a way to construct a minimal ($\mathfrak{S}_{m+1}$-equivariant) free resolution of I_{m+1} from a given minimal ($\mathfrak{S}_m$-equivariant) free resolution of I_m.

Combinatorics of partitions. The set Λ of all partitions has a natural poset structure if we define its order by $\lambda \leq \mu$ if x^λ divides x^μ. We say that a subset $P \subset \Lambda$ is an ideal (or sometimes called a filter) of Λ if for any $\lambda \in P$ and $\mu > \lambda$ one has $\mu \in P$. Then, for any symmetric monomial ideal $\mathcal{I} \subset S_\infty$, the set $P(\mathcal{I}) = \{\lambda \in \Lambda : x^\lambda \in \mathcal{I}\}$ is an ideal of Λ. It is not hard to see that the assignment $\mathcal{I} \to P(\mathcal{I})$ gives a one to one correspondent between symmetric monomial ideals in S_∞ and ideals in Λ (see [BdAG+]). This natural correspondence suggests the following problem.

Problem 5.4. Study relations between algebraic properties of symmetric monomial ideals in S_∞ (or S_n) and combinatorial properties of ideals in Λ.

As we remarked in the introduction, Raicu [Ra] find a combinatorial formula of the projective dimension and the regularity of a symmetric monomial ideal I in S_n. It would be interesting to find a combinatorial way to determine $\beta_{i,j}(I)$ (or more strongly $\Bbbk[\mathfrak{S}_n]$-module structure of $\mathrm{Tor}_i(I, \Bbbk)_j$) from combinatorial information of $P(I)$.

We note that the graded Betti numbers may depend on the characteristic of $\Bbbk$. For example, if $I \subset S_6$ is generated by the following 10 partitions

$$(5,4,4,2,2,1),(5,4,4,3,1,1),(5,5,3,3,1,1),(5,5,3,3,2),(5,5,4,2,2),$$
$$(6,4,3,2,2,1),(6,4,3,3,2),(6,4,4,3,1),(6,5,3,2,1,1),(6,5,4,2,1),$$

then $\Delta^I_{(6,5,4,3,2,1)}$ is a triangulation of $\mathbb{R}P^2$ and $\beta_{2,(6,5,4,3,2,1)}(I)$ is 0 if $\mathrm{char}(\Bbbk) \neq 2$ and 1 if $\mathrm{char}(\Bbbk) = 2$. On the other hand, the result of Raicu tells that projective dimension and regularity do not depend on the characteristic of the ground field.

Generalization. The study of symmetric monomial ideals in S_∞ is a very special case of a more general setting studied by Le, Nagel, Nguyen and Römer [LNNR1, LNNR2], who consider homogeneous ideals in $R = \Bbbk[x_{i,j} : 1 \leq i \leq c, j \geq 1]$ fixed by the action of the monoid $\mathrm{inc}(\mathbb{N}) = \{\sigma : \mathbb{N} \to \mathbb{N} \mid \sigma(i+1) > \sigma(i) \text{ for all } i\}$. It would be an attractive problem to find generalizations of the results in this paper to such a more general setting.

Question 5.5. Can we find a (possibly weaker) version of Theorem 1.1 which holds for all homogeneous ideals, ideals in R, or ideals fixed by the $\mathrm{inc}(\mathbb{N})$-action?

The case when $\mathcal{I}$ is not a monomial ideal or when $\mathcal{I}$ is not fixed by $\mathfrak{S}_\infty$-action seems more complicated. Below we write a few example showing that the situation is not as simple as the case of monomial ideals in S_∞.

Example 5.6. It was pointed out by Hop Nguyen that the statement of Corollary 1.3 (and therefore Theorem 1.1 also) does not hold for $\mathrm{inc}(\mathbb{N})$-invariant monomial ideals. Indeed, if

$$\mathcal{I} = (\sigma(x_1^2x_2) : \sigma \in \mathrm{inc}(\mathbb{N})) + (\sigma(x_1^3) : \sigma \in \mathrm{inc}(\mathbb{N}))$$

and $I_n = \mathcal{I} \cap S_n$, then $x_1x_2\cdots x_{k-1}x_k^2$ is a socle element of S_n/I_n for $k = 1,2,\dots,n$. This tells that $|\{j : \beta_{n-1,j}(I_n) \neq 0\}| \geq n$, so even Corollary 1.3 does not hold for $\mathrm{inc}(\mathbb{N})$-invariant monomial ideals.

Example 5.7. There are symmetric homogeneous ideals whose Betti table has similar shape as ideals in Example 5.6. Let $n \geq k$ be positive integers and let

$$I^{\mathrm{Vd}}_{n,k} = \Big(\textstyle\prod_{1 \leq p < q \leq k}(x_{i_p} - x_{i_q}) : 1 \leq i_1 < \cdots < i_k \leq n\Big) \subset S_n.$$

The ideal $I^{\mathrm{Vd}}_{n,k}$ is fixed by an action of $\mathfrak{S}_n$, so we get a chain of symmetric ideals

$$\cdots \subset I^{\mathrm{Vd}}_{n-1,k} \subset I^{\mathrm{Vd}}_{n,k} \subset I^{\mathrm{Vd}}_{n+1,k} \subset \cdots.$$

The minimal free resolution of $I^{\mathrm{Vd}}_{n,k}$ was given by Yanagawa and Watanabe [YW]. Their result tells $\lim_{n\to\infty} |\{j : \beta_{n-k-1,j}(I^{\mathrm{Vd}}_{n,k}) \neq 0\}| = \infty$. See [YW, Theorem 2.2 and Corollary 2.3].

Example 5.8. Consider the symmetric ideal

$$I_n = (x_i^2 : i = 1,2,\dots,n) + (x_ix_j - x_kx_l : 1 \leq i < j \leq n, 1 \leq k < l \leq n) \subset S_n.$$

The ideal is the defining ideal of the graded Möbius algebra for a rank 1 simple matroid [MN, HW]. The ideal I_n is clearly fixed by the action of $\mathfrak{S}_n$, and it is not hard to see that S_n/I_n is an Artinian Gorenstein algebra with Hilbert function $(1,n,1)$. So, for any integer $n \geq 2$, the Betti table of I_n has the following form.

	0	1	2	$\cdots$	$\cdots$	$n-2$	$n-1$
2	$*$	$*$	$*$	$\cdots$	$\cdots$	$*$	
3							1

The shape of the Betti table is clearly not a union of line segments.

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