# TIGHT COMBINATORIAL MANIFOLDS AND GRADED BETTI NUMBERS

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Abstract. In this paper, we study the conjecture of Kühnel and Lutz, who state that a combinatorial triangulation of the product of two spheres $\mathbb{S}^i \times \mathbb{S}^j$ with $j \geq i$ is tight if and only if it has exactly $i+2j+4$ vertices. To approach this conjecture, we use graded Betti numbers of Stanley–Reisner rings. By using recent results on graded Betti numbers, we prove that the only if part of the conjecture holds when $j > 2i$ and that the if part of the conjecture holds for triangulations all whose vertex links are simplicial polytopes. We also apply this algebraic approach to obtain lower bounds on the numbers of vertices and edges of triangulations of manifolds and pseudomanifolds.

## 1. Introduction

The tightness of simplicial complexes is an important property which appears in the study of triangulations of topological manifolds. Let $\Delta$ be a (finite abstract) simplicial complex on the vertex set $V$. For $W \subset V$, we write $\Delta_W = \{F \in \Delta : F \subset W\}$ for the induced subcomplex of $\Delta$ on $W$. The simplicial complex $\Delta$ is said to be $\mathbb{F}$-*tight* if the natural map on the homologies

$$H_i(\Delta_W; \mathbb{F}) \to H_i(\Delta; \mathbb{F})$$

induced by the inclusion is injective for all $i$ and all $W \subset V$, where $\mathbb{F}$ is a field and where $H_i(\Delta; \mathbb{F})$ is the $i$th homology group of $\Delta$ with coefficients in $\mathbb{F}$. This concept comes from differential geometry but has interesting connections to topology, convex geometry and combinatorics. We refer the readers to [Kü] for the background and motivation on tight triangulations.

In the combinatorial study of triangulations of manifolds, vertex minimal triangulations are important research objects (see [Lu] for a survey). One interesting combinatorial feature of the tightness is that it often appears in vertex minimal triangulations. A *combinatorial $d$-manifold* is a simplicial complex such that every vertex link is PL homeomorphic to the boundary of a $d$-simplex. A combinatorial manifold whose geometric realization is homeomorphic to a closed manifold $M$ is called a *combinatorial triangulation* of $M$. Kühnel and Lutz conjectured that every tight combinatorial triangulation is vertex minimal [KL, Conjecture 1.3]. Moreover, for combinatorial triangulations of the product of two spheres, they proposed the following more precise conjecture [KL, Conjecture 1.5].

**Conjecture 1.1** (Kühnel–Lutz)**.** A combinatorial triangulation of $\mathbb{S}^i \times \mathbb{S}^j$ with $j \geq i$ is $\mathbb{F}$-tight if and only if it has exactly $i + 2j + 4$ vertices.

Since a combinatorial triangulation of $\mathbb{S}^i \times \mathbb{S}^j$ has at least $i + 2j + 4$ vertices by the result of Brehm and Kühnel [BK, Corollary 1], the only if part of the conjecture implies that tight combinatorial triangulations of the product of two spheres are vertex minimal (if they exist). Conjecture 1.1 is known to be true when $i = j$ [Kü, Corollary 4.7] and when $i = 1$ (see [DM, Corollary 4.4] and [Kü, Theorem 5.3]). In this paper, we give new partial affirmative answers to the conjecture.

Let $\Delta$ be a connected combinatorial $d$-manifold. We say that $\Delta$ is $\mathbb{F}$-*orientable* if $H_d(\Delta;\mathbb{F}) \cong \mathbb{F}$. Also, we say that $\Delta$ is *locally polytopal* if every vertex link of $\Delta$ is the boundary of a simplicial $d$-polytope. For a simplicial complex $\Delta$, let $b_i(\Delta;\mathbb{F}) = \dim_{\mathbb{F}} H_i(\Delta;\mathbb{F})$. The main result of this paper is the following.

**Theorem 1.2.** *Let $\Delta$ be a connected combinatorial $d$-manifold with $n$ vertices and let $r < \frac{d}{2}$ be a positive integer.*

(i) *Suppose that $\Delta$ is $\mathbb{Q}$-orientable and locally polytopal. If*

$$\binom{n-d-2+r}{r+1} = \binom{d+2}{r+1} b_r(\Delta;\mathbb{Q}),$$

*then $\Delta$ is $\mathbb{Q}$-tight.*

(ii) *Suppose $r \leq \frac{d-1}{3}$ and $b_i(\Delta;\mathbb{F}) = 0$ for $i \notin \{0, r, d-r, d\}$. If $\Delta$ is $\mathbb{F}$-tight, then*

$$\binom{n-d-2+r}{r+1} = \binom{d+2}{r+1} b_r(\Delta;\mathbb{F}).$$

The theorem says that, under certain assumptions, tightness only depends on the number of vertices and Betti numbers. Note that in the special case when $r = 1$ the above theorem was proved in [BDSS, DM, Ef] without local polytopality assumption.

By considering the special case of Theorem 1.2 when $b_r(\Delta;\mathbb{F}) = 1$, we obtain the following partial answers to Conjecture 1.1.

**Corollary 1.3.** *Let $\Delta$ be an $n$ vertex combinatorial triangulation of $\mathbb{S}^i \times \mathbb{S}^j$ with $j > i$. Then*

(i) *if $\Delta$ is locally polytopal and $n = i + 2j + 4$, then $\Delta$ is $\mathbb{Q}$-tight.*

(ii) *if $j > 2i$ and $\Delta$ is $\mathbb{F}$-tight, then $n = i + 2j + 4$.*

In particular, the above corollary shows that tight combinatorial triangulations of $\mathbb{S}^i \times \mathbb{S}^j$ are vertex minimal when $j > 2i$.

Theorem 1.2 is closely related to the following conjecture of Kühnel [Lu, Conjecture 18] on lower bounds on the number of vertices of combinatorial manifolds.

**Conjecture 1.4** (Kühnel)**.** Let $\Delta$ be a connected combinatorial $d$-manifold with $n$ vertices. Then

$$\binom{n-d-2+r}{r+1} \geq \binom{d+2}{r+1} b_r(\Delta;\mathbb{F}) \tag{1}$$

for $1 \leq r \leq \frac{d-1}{2}$. Moreover, if $d$ is even, then

$$\binom{n-\frac{d}{2}-2}{\frac{d}{2}+1} \geq \binom{d+2}{\frac{d}{2}+1} \frac{1}{2} b_{\frac{d}{2}}(\Delta;\mathbb{F}). \tag{2}$$

Indeed, Theorem 1.2 discusses the equality case of (1). Moreover, it is known that if an $\mathbb{F}$-orientable combinatorial manifold satisfies the equality in (2), then it is $\mathbb{F}$-tight (see [Kü, Theorem 5.3] and [NS2, Theorem 4.3]).

We apply the methods used in the proof of Theorem 1.2 to the above conjecture. Conjecture 1.4 was proved by Novik and Swartz [NS1, NS2] for several cases. Assuming $\mathbb{F}$-orientability, they proved the inequality (2), the inequality (1) when $r = 1$; they also proved the conjecture for locally polytopal combinatorial manifolds. In Theorems 5.1 and 5.2, we extend their results to non-orientable manifolds. By using the same technique, we also give lower bound on the number of edges of normal pseudomanifolds. One of the fundamental results in the study of face numbers of simplicial complexes is the lower bound theorem, proved by Barnette [Bar1, Bar2] for simplicial polytopes and extended to normal pseudomanifolds in [Fo, Ka, Ta]. (See Section 5 for the definitions of normal puseudomanifolds, stacked simplicial spheres and stacked simplicial manifolds.)

**Theorem 1.5** (Lower bound theorem)**.** *If $\Delta$ is a normal pseudomanifold of dimension $d \geq 2$ with $n$ vertices, then the number of edges of $\Delta$ is larger than or equal to $(d+1)n - \binom{d+2}{2}$. Moreover, if $d \geq 3$, then the equality holds if and only if $\Delta$ is a stacked simplicial $d$-sphere.*

Kalai [Ka] conjectured that the above lower bound can be refined to $(d+1)n + \binom{d+2}{2}(b_1(\Delta;\mathbb{Q})-1)$ for triangulations of closed manifolds of dimension $\geq 3$, and this conjecture was later solved by Novik and Swartz [NS1, Theorem 5.2]. In Theorem 5.3, we prove the following refinement of the lower bound theorem.

**Theorem 1.6.** *If $\Delta$ is a normal pseudomanifold of dimension $d \geq 3$ with $n$ vertices, then the number of edges of $\Delta$ is larger than or equal to $(d+1)n+\binom{d+2}{2}(b_1(\Delta;\mathbb{F})-1)$. Moreover, the equality holds if and only if $\Delta$ is a stacked simplicial $d$-manifold.*

The crucial idea to prove the results is the use of commutative algebra. We first show that, by the characterization of the tightness in terms of Betti numbers of induced subcomplexes of vertex links given by Bagchi and Datta [Bag1, BD], the tightness can be studied by using graded Betti numbers of Stanley–Reisner rings, which are well-studied algebraic invariant in commutative algebra. Then we prove our results with the help of recent theorems on graded Betti numbers.

This paper is organized as follows: In Section 2, we explain that the tightness can be characterized by using graded Betti numbers of Stanley–Reisner rings. In Section 3, we prove Theorem 1.2(i) by using upper bounds on graded Betti numbers of simplicial polytopes given by Migliore and Nagel. In Section 4, we prove Theorem 1.2(ii) based on a recent result on the subadditivity condition for syzygies of monomial ideals given by Herzog and Srinivasan. In Section 5, we study lower bounds on the numbers of vertices and edges using upper bounds on graded Betti numbers. In Section 6, we present some open questions.

## 2. Tightness and graded Betti numbers

In this section, we explain a relation between the tightness and graded Betti numbers. We first introduce necessary notations. Let $\Delta$ be a simplicial complex on the vertex set $V$. Thus $\Delta$ is a collection of subsets of $V$ satisfying

(i) $F \in \Delta$ and $G \subset F$ imply $G \in \Delta$.
(ii) $\{v\} \in \Delta$ for any $v \in V$.

Elements of $\Delta$ are called *faces*. The *dimension* of a face is its cardinality minus 1 and the *dimension* of $\Delta$ is the maximal dimension of its faces. Faces of dimension 0 are called *vertices* of $\Delta$. We denote by $f_i(\Delta)$ the number of $i$-dimensional faces of $\Delta$. For a face $F \in \Delta$, the simplicial complex

$$\mathrm{lk}_\Delta(F) = \{G \in \Delta : F \cap G = \emptyset, F \cup G \in \Delta\}$$

is called the *link* of $F$ in $\Delta$. For simplicity, we write $\mathrm{lk}_\Delta(v) = \mathrm{lk}_\Delta(\{v\})$. We say that $\Delta$ is a triangulation of a topological space $X$ if its geometric realization is homeomorphic to $X$.

Next, we recall a criterion for tightness in terms of Betti numbers of induced subcomplexes of vertex links. Let $\Delta$ be a simplicial complex on $V$. We write $\widetilde{H}_i(\Delta;\mathbb{F})$ for the $i$th reduced homology group of $\Delta$ with coefficients in $\mathbb{F}$ and write $\widetilde{b}_i(\Delta;\mathbb{F}) = \dim_\mathbb{F} \widetilde{H}_i(\Delta;\mathbb{F})$, where we define $\widetilde{b}_0(\{\emptyset\};\mathbb{F}) = 0$ and $\widetilde{b}_{-1}(\{\emptyset\};\mathbb{F}) = 1$. The $j$th *$\sigma$-number* of $\Delta$ (over $\mathbb{F}$) is the number

$$\sigma_j(\Delta;\mathbb{F}) = \sum_{W \subset V} \frac{1}{\binom{\#V}{\#W}} \widetilde{b}_j(\Delta_W;\mathbb{F})$$

for $j = -1, 0, 1, \dots, \dim \Delta$, where $\#X$ denotes the cardinality of a finite set $X$. Note that $\sigma_{-1}(\Delta;\mathbb{F}) = \widetilde{b}_{-1}(\{\emptyset\};\mathbb{F}) = 1$ and that $\sigma_0$ in this paper is $\sigma_0$ in [Bag1, BD, BDSS] plus 1 since we assume that $\widetilde{b}_0(\{\emptyset\};\mathbb{F}) = 0$. We also define

$$\mu_j(\Delta;\mathbb{F}) = \sum_{v \in V} \frac{\sigma_{j-1}(\mathrm{lk}_\Delta(v);\mathbb{F})}{f_0(\mathrm{lk}_\Delta(v)) + 1}$$

for $j = 0, 1, \dots, \dim \Delta$. The following result was first proved by Bagchi and Datta [BD] for 2-neighborly simplicial complexes, and was extended to all simplicial complexes by Bagchi [Bag1, Theorems 1.6 and 1.7].

**Theorem 2.1** (Bagchi)**.** *Let $\Delta$ be a simplicial complex. Then*

- (i) $b_j(\Delta;\mathbb{F}) \leq \mu_j(\Delta;\mathbb{F})$ *for all $j$.*
- (ii) *$\Delta$ is $\mathbb{F}$-tight if and only if $b_j(\Delta;\mathbb{F}) = \mu_j(\Delta;\mathbb{F})$ for all $j$.*
- (iii) *if $\Delta$ is a triangulation of a $d$-sphere, then $\sigma_{j-1}(\Delta;\mathbb{F}) = \sigma_{d-j}(\Delta;\mathbb{F})$ for all $j$.*
- (iv) *if $\Delta$ is a triangulation of a closed $d$-manifold, then $\mu_j(\Delta;\mathbb{F}) = \mu_{d-j}(\Delta;\mathbb{F})$ for all $j$.*
- (v) $\sum_{k=0}^{j}(-1)^{j-k} b_k(\Delta;\mathbb{F}) \leq \sum_{k=0}^{j}(-1)^{j-k}\mu_k(\Delta;\mathbb{F})$ *for all $j$.*

A simplicial complex $\Delta$ on $V$ is said to be *$j$-neighborly* if $\Delta$ contains all subsets of $V$ of cardinality $\leq j$. If $\Delta$ is a connected simplicial complex, then $\mu_0(\Delta;\mathbb{F}) = b_0(\Delta;\mathbb{F})$ if and only if $\Delta$ is 2-neighborly. Also, if $\Delta$ is a triangulation of a connected closed $d$-manifold, then $\mu_d(\Delta;\mathbb{F}) = b_d(\Delta;\mathbb{F})$ if and only if $\Delta$ is 2-neighborly and $\mathbb{F}$-orientable. We will use the following special case of Theorem 2.1(ii) to check the tightness.

**Lemma 2.2.** *Let $\Delta$ be a 2-neighborly $\mathbb{F}$-orientable combinatorial $d$-manifold. Then $\Delta$ is $\mathbb{F}$-tight if and only if $b_i(\Delta;\mathbb{F}) = \mu_i(\Delta;\mathbb{F})$ for all $1 \leq i \leq d-1$.*

Next, we introduce graded Betti numbers. Let $S = \mathbb{F}[x_1, \dots, x_n]$ be the graded polynomial ring with $\deg x_i = 1$ for all $i$. For a graded $S$-module $M$, we write $M_k = \{u \in M : \deg u = k\} \cup \{0\}$ for its graded component of degree $k$. Let $I \subset S$ be a homogeneous ideal. The integer

$$\beta^S_{i,j}(S/I) = \dim_\mathbb{F} \mathrm{Tor}^S_i(S/I,\mathbb{F})_j$$

is called the $(i,j)$*th graded Betti number* of $S/I$, where we identify $S/(x_1,\dots,x_n)$ and $\mathbb{F}$. Similarly, the integers $\beta^S_{i,j}(I)=\dim_{\mathbb{F}}\mathrm{Tor}^S_i(I,\mathbb{F})_j$ are called the graded Betti numbers of $I$. By the short exact sequence $0\to I\to S\to S/I\to 0$, $\beta^S_{i,j}(I)$ and $\beta^S_{i,j}(S/I)$ are related by

$$\beta^S_{i,j}(I)=\beta^S_{i+1,j}(S/I)$$

for all $i,j\geq 0$.

For a simplicial complex $\Delta$ on the vertex set $[n]=\{1,2,\dots,n\}$, its *Stanley–Reisner ideal* $I_\Delta\subset S$ is the ideal generated by the squarefree monomials $x_F=\Pi_{i\in F}x_i$ with $F\notin\Delta$. Thus

$$I_\Delta=(x_F : F\subset[n],\ F\notin\Delta).$$

The ring $\mathbb{F}[\Delta]=S/I_\Delta$ is called the *Stanley–Reisner ring* of $\Delta$. For a Stanley–Reisner ring $\mathbb{F}[\Delta]=S/I_\Delta$, we write $\beta_{i,j}(\mathbb{F}[\Delta])=\beta^S_{i,j}(S/I_\Delta)$. The following result is known as Hochster's formula (for graded Betti numbers). See [BH, Theorem 5.5.1].

**Theorem 2.3** (Hochster)**.** *Let $\Delta$ be a simplicial complex on $[n]$. Then*

$$\beta_{i,i+j}(\mathbb{F}[\Delta])=\sum_{W\subset[n],\ \#W=i+j}\widetilde{b}_{j-1}(\Delta_W;\mathbb{F})$$

*for all $i,j\geq 0$.*

As an immediate consequence of Hochster's formula, we have the following expression of $\sigma$-numbers

$$\sigma_{j-1}(\Delta;\mathbb{F})=\sum_{k=0}^{n}\left(\sum_{W\subset[n],\ \#W=k}\frac{\widetilde{b}_{j-1}(\Delta_W;\mathbb{F})}{\binom{n}{k}}\right)=\sum_{k=j}^{n}\frac{\beta_{k-j,(k-j)+j}(\mathbb{F}[\Delta])}{\binom{n}{k}}. \tag{3}$$

Note that $\widetilde{b}_{j-1}(\Delta_W)=0$ if $\#W<j$. The above formula and Theorem 2.1 show that we may study the tightness of simplicial complexes algebraically. In the rest of this paper, we study the tightness by using graded Betti numbers.

## 3. Locally polytopal combinatorial manifolds

In this section, we prove Theorem 1.2(i). For a simplicial complex $\Delta$ of dimension $d-1$, its *h-vector* $h(\Delta)=(h_0(\Delta),h_1(\Delta),\dots,h_d(\Delta))$ is the sequence defined by

$$h_i(\Delta)=\sum_{k=0}^{i}(-1)^{i-k}\binom{d-k}{i-k}f_{k-1}(\Delta),$$

where $f_{-1}(\Delta)=1$. Throughout the paper, we regard a simplicial $d$-polytope $P$ as a simplicial complex of dimension $d-1$ by identifying $P$ with its boundary complex. To prove Theorem 1.2(i), we first prove the following bounds on $\sigma$-numbers.

**Theorem 3.1.** *Let $P$ be a simplicial $d$-polytope with $n$ vertices and let $r<\frac{d}{2}$ be a positive integer. Then*

$$\sigma_{r-1}(P;\mathbb{Q})\leq\frac{1}{\binom{d+2}{r+1}}\binom{n-d-1+r}{r+1}.$$

*If the equality holds, then $P$ is $r$-neighborly and $h_r(P)=h_{r+1}(P)$.*

Note that Theorem 3.1 for $d = 3$ is a recent result of Burton, Datta, Singh and Spreer [BDSS, Theorem 1.1]. To prove Theorem 3.1, we consider upper bounds on graded Betti numbers. Note that, by (3), upper bounds on graded Betti numbers of Stanley–Reisner rings yield those on $\sigma$-numbers.

For a homogenous ideal $I$ of $S = \mathbb{F}[x_1, \dots, x_n]$, the *Hilbert function* of $S/I$ is the function $\mathrm{Hilb}(S/I, -) : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined by

$$\mathrm{Hilb}(S/I, k) = \dim_{\mathbb{F}} (S/I)_k$$

for all $k \in \mathbb{Z}_{\geq 0}$. Let $>_{\mathrm{lex}}$ be the lexicographic order on $S$ with $x_1 >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} x_n$. Thus, $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} >_{\mathrm{lex}} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ if the leftmost non-zero entry of $(a_1 - b_1, \dots, a_n - b_n)$ is positive. A *monomial ideal* of $S$ is an ideal generated by monomials in $S$. A *lex ideal* is a monomial ideal $L \subset S$ which satisfies that, for any monomials $u, v$ of the same degree, $u \in L$ and $v >_{\mathrm{lex}} u$ imply $v \in L$. For a monomial ideal $I$, we write $G(I)$ for the unique set of minimal monomial generators of $I$. For a monomial $u \in S$, let $\max(u)$ be the largest integer $i$ such that $x_i$ divides $u$. The following results are well-known in commutative algebra.

**Theorem 3.2.** *Let $I \subset S$ be a homogenous ideal. Then*

(i) (Macaulay [Ma]) *there is a unique lex ideal, denoted $I^{\mathrm{lex}}$, such that $S/I$ and $S/I^{\mathrm{lex}}$ have the same Hilbert function.*
(ii) (Bigatti [Bi], Hulett [Hu], Pardue [Pa]) $\beta_{i,j}^S(S/I) \leq \beta_{i,j}^S(S/I^{\mathrm{lex}})$ *for all $i, j$.*

**Theorem 3.3** (Eliahou–Kervaire [EK])**.** *Let $L \subset S$ be a lex ideal. Then*

$$\beta_{i,i+j}^S(S/L) = \sum_{u \in G(L),\ \deg u = j+1} \binom{\max(u) - 1}{i - 1}$$

*for all $i \geq 1$ and $j \geq 0$.*

Let $\mathfrak{m} = (x_1, \dots, x_n)$ be the homogenous maximal ideal of $S$. The next lemma is an easy consequence of Theorems 3.2 and 3.3.

**Lemma 3.4.** *Let $r$ be a positive integer.*

(i) $\beta_{i,i+r}^S(S/\mathfrak{m}^{r+1}) = \binom{i-1+r}{r}\binom{n+r}{i+r}$ *for all $i \geq 0$.*
(ii) *For any homogeneous ideal $I \subset S$, we have*

$$\beta_{i,i+r}^S(S/I) \leq \beta_{i,i+r}^S(S/\mathfrak{m}^{r+1})$$

*for all $i$. Moreover, the equality holds for all $i$ if and only if $I = \mathfrak{m}^{r+1}$.*

*Proof.* Observe that $G(\mathfrak{m}^{r+1})$ is the set of all degree $r+1$ monomials in $S$ and that the number of degree $r+1$ monomials $u \in S$ with $\max(u) = \ell$ is $\binom{\ell-1+r}{r}$ since it is equal to the number of degree $r$ monomials in $\mathbb{F}[x_1, \dots, x_\ell]$. Then Theorem 3.3 says

$$\beta_{i,i+r}^S(S/\mathfrak{m}^{r+1}) = \sum_{\ell=i}^{n} \binom{\ell - 1 + r}{r}\binom{\ell - 1}{i - 1}$$

for all $i \geq 1$. Now the statement (i) follows from the next computation

$$\begin{aligned}\sum_{\ell=i}^{n}\binom{\ell-1+r}{r}\binom{\ell-1}{i-1} &= \frac{1}{r!(i-1)!}\sum_{\ell=i}^{n}\frac{(\ell-1+r)!}{(\ell-i)!}\\ &= \binom{i-1+r}{r}\sum_{\ell=i}^{n}\binom{\ell-1+r}{i-1+r}\\ &= \binom{i-1+r}{r}\binom{n+r}{i+r}.\end{aligned}$$

Next, we prove (ii). Since $G(\mathfrak{m}^{r+1})$ is the set of all degree $r+1$ monomials, Theorems 3.2 and 3.3 show

$$\beta^S_{i,i+r}(S/I) \leq \beta^S_{i,i+r}(S/I^{\mathrm{lex}}) \leq \beta^S_{i,i+r}(S/\mathfrak{m}^{r+1})$$

for all $i$. Also, since $\beta^S_{1,r+1}(S/I) = \beta^S_{0,r+1}(I)$ is the number of degree $r+1$ elements in a minimal generating set of $I$, $\beta^S_{1,r+1}(S/I) = \beta^S_{1,r+1}(S/\mathfrak{m}^{r+1})$ implies that $I$ has $\dim_{\mathbb{F}} S_{r+1}$ generators of degree $r+1$, which guarantees $I = \mathfrak{m}^{r+1}$. □

Next, we recall a result of Migliore and Nagel which connects the graded Betti numbers of lex ideals and those of Stanley–Reisner rings of simplicial polytopes. A sequence $\mathbf{h} = (h_0, h_1, \ldots, h_s) \in \mathbb{Z}^{s+1}_{\geq 0}$ is called an *$M$-vector* if there is a homogeneous ideal $I \subset \mathbb{F}[x_1, \ldots, x_{h_1}]$ such that $h_k = \dim_{\mathbb{F}}(\mathbb{F}[x_1, \ldots, x_{h_1}]/I)_k$ for all $k = 0, 1, \ldots, s$. By Macaulay's theorem (Theorem 3.2(i)), if $\mathbf{h} = (h_0, h_1, \ldots, h_s)$ is an $M$-vector, then there is a unique lex ideal $L$ of $R = \mathbb{Q}[x_1, \ldots, x_{h_1}]$ such that $\mathrm{Hilb}(R/L, k)$ is $h_k$ for $k \leq s$ and is zero for $k > s$. We write $L_{\mathbf{h}}$ for this unique lex ideal of $R = \mathbb{Q}[x_1, \ldots, x_{h_1}]$.

The $g$-theorem [St, III Theorem 1.1] says that if $P$ is a simplicial $d$-polytope, then its *$g$-vector*

$$g(P) = \big(h_0(P), h_1(P) - h_0(P), \ldots, h_{\lfloor \frac{d}{2} \rfloor}(P) - h_{\lfloor \frac{d}{2} \rfloor - 1}(P)\big)$$

is an $M$-vector, where $\lfloor a \rfloor$ denotes the integer part of $a \in \mathbb{Q}$. Observe $h_1(P) - h_0(P) = f_0(P) - d - 1$. The following result was proved by Migliore and Nagel [MiN, Theorem 9.6].

**Theorem 3.5** (Migliore–Nagel)**.** *Let $P$ be a simplicial $d$-polytope with $n$ vertices, $\mathbf{g} = g(P)$ and $R = \mathbb{Q}[x_1, \ldots, x_{n-d-1}]$. Then, for all $i \geq 0$ and $j \geq 0$,*

$$\beta_{i,i+j}(\mathbb{Q}[P]) \leq \begin{cases} \beta^R_{i,i+j}(R/L_{\mathbf{g}}), & \text{if } j < \frac{d}{2},\\ \beta^R_{i,i+j}(R/L_{\mathbf{g}}) + \beta^R_{n-d-i,n-i-j}(R/L_{\mathbf{g}}), & \text{if } j = \frac{d}{2},\\ \beta^R_{n-d-i,n-i-j}(R/L_{\mathbf{g}}), & \text{if } j > \frac{d}{2}.\end{cases}$$

We are now ready to prove Theorem 3.1. Before proving it, we note the following technical equation.

**Lemma 3.6.** *For positive integers $n, d$ and $r$ with $n > d \geq r$, one has*

$$\sum_{k=1}^{n}\frac{1}{\binom{n}{k}}\binom{k-1}{r}\binom{n-d-1+r}{k} = \frac{1}{\binom{d+2}{r+1}}\binom{n-d-1+r}{r+1}.$$

*Proof.* The desired formula follows from the next computations

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \binom{k-1}{r}\binom{n-d-1+r}{k}\\
&= \frac{(n-d-1+r)!}{n!} \sum_{k=r+1}^{n-d-1+r} \binom{k-1}{r} \frac{(n-k)!}{(n-d-1+r-k)!}\\
&= \frac{(n-d-1+r)!(d+1-r)!}{n!} \sum_{k=r+1}^{n-d-1+r} \binom{k-1}{r}\binom{n-k}{d+1-r}\\
&\overset{(\star)}{=} \frac{(n-d-1+r)!(d+1-r)!}{n!}\binom{n}{d+2}\\
&= \frac{1}{\binom{d+2}{r+1}}\binom{n-d-1+r}{r+1},
\end{aligned}$$

where $(\star)$ follows from the partition

$$\begin{aligned}
&\{F \subset [n] : \#F = d+2\}\\
&= \biguplus_{k=r+1}^{n-d-1+r} \big\{F \cup \{k\} \cup G : \max(F) < k < \min(G),\ \#F = r,\ \#G = d+1-r\big\}.
\end{aligned}$$

□

*Proof of Theorem 3.1.* Let $\mathbf{g} = g(P)$ and $R = \mathbb{Q}[x_1, \dots, x_{n-d-1}]$. By Lemma 3.4 and Theorem 3.5,

$$\beta_{i,i+r}(\mathbb{Q}[P]) \le \beta^R_{i,i+r}(R/L_{\mathbf{g}}) \le \binom{i-1+r}{r}\binom{n-d-1+r}{i+r} \tag{4}$$

for all $i$. The above inequality and (3) say

$$\sigma_{r-1}(P;\mathbb{Q}) = \sum_{k=r}^{n} \frac{1}{\binom{n}{k}} \beta_{k-r,(k-r)+r}(\mathbb{Q}[P]) \le \sum_{k=r}^{n} \frac{1}{\binom{n}{k}} \binom{k-1}{r}\binom{n-d-1+r}{k}.$$

Then the desired inequality follows from Lemma 3.6.

Suppose $\binom{d+2}{r+1}\sigma_{r-1}(P;\mathbb{Q}) = \binom{n-d-1+r}{r+1}$. Then we have the equality in (4) for all $i$. Thus $L_{\mathbf{g}} = (x_1, \dots, x_{n-d-1})^{r+1}$ by Lemma 3.4. This implies $h_i(P) = \binom{n-d-1+i}{i}$ for $i \le r$ and $h_r(P) = h_{r+1}(P)$, where when $r = \frac{d-1}{2}$ the equation $h_r(P) = h_{r+1}(P)$ follows from the Dehn–Sommerville equations (see [Zi, Theorem 8.21]). Since the former condition is equivalent to saying that $f_{i-1}(P) = \binom{n}{i}$ for all $i \le r$ (see [Zi, Lemma 8.26]), $P$ is $r$-neighborly. □

Let $P$ be a simplicial $d$-polytope satisfying the equality in Theorem 3.1. Then the $h$-vector of $P$ only depends on $n, d$ and $r$. Indeed, the Dehn–Sommerville equations and the $r$-neighborly property say $h_{d-i}(P) = h_i(P) = \binom{n-d-1+i}{i}$ for $i \le r$. Also $h_r(P) = h_{r+1}(P)$ implies $h_r(P) = \cdots = h_{d-r}(P)$ since $g(P)$ is an $M$-vector. The graded Betti numbers of simplicial polytopes having such an $h$-vector were computed in [MiN, Corollary 8.14 and Corollary 9.8] when $r < \frac{d-1}{2}$.

**Theorem 3.7** (Migliore–Nagel). *Let $P$ a simplicial $d$-polytope with $n$ vertices and let $r < \frac{d-1}{2}$ be a positive integer. If $P$ is $r$-neighborly and $h_r(P) = h_{r+1}(P)$, then*

(i) $\beta_{i,i+j}(\mathbb{Q}[P]) = 0$ *for all* $i \geq 0$ *and* $j \notin \{0, r, d-r, d\}$.

(ii) $\beta_{i,i+r}(\mathbb{Q}[P]) = \beta_{n-d-i,(n-d-i)+d-r}(\mathbb{Q}[P]) = \binom{i-1+r}{r}\binom{n-d-1+r}{i+r}$ *for all* $i \geq 0$.

The above result, Theorem 3.1, and (3) imply the following.

**Corollary 3.8.** *Let $P$ a simplicial $d$-polytope with $n$ vertices and $r < \frac{d}{2}$ a positive integer. If $\sigma_{r-1}(P;\mathbb{Q}) = \frac{1}{\binom{d+2}{r+1}}\binom{n-d-1+r}{r+1}$, then $\sigma_{k-1}(P;\mathbb{Q}) = 0$ for $k \notin \{0, r, d-r, d\}$.*

Note that, when $r = \frac{d-1}{2}$, the above corollary follows from Theorem 2.1(iii) since the $r$-neighborly property implies $\sigma_{k-1}(P) = 0$ for $1 \leq k < r$.

**Remark 3.9.** A recent result of Bagchi [Bag2, Lemma 3] and the generalized lower bound theorem (see [MuN1]) prove that, when $r < \frac{d-1}{2}$, a simplicial $d$-polytope $P$ with $h_r(P) = h_{r+1}(P)$ satisfies $\sigma_{k-1}(P) = 0$ for $r < k < d - r$. This result gives an alternative proof of Corollary 3.8.

We also need the equality case of the following result of Novik and Swartz [NS2, Theorem 4.3]. (We will discuss an extension of this result to non-orientable manifolds later in Section 5.)

**Theorem 3.10** (Novik–Swartz). *Let $\Delta$ be a connected $\mathbb{Q}$-orientable locally polytopal combinatorial $d$-manifold with $n$ vertices and let $r < \frac{d}{2}$ be a positive integer. Then $\binom{n-d-2+r}{r+1} \geq \binom{d+2}{r+1} b_r(\Delta;\mathbb{Q})$. If the equality holds then $\Delta$ is $(r+1)$-neighborly.*

We now prove Theorem 1.2(i).

*Proof of Theorem 1.2(i).* The assumption and Theorem 3.10 say that $\Delta$ is $(r+1)$-neighborly. Then by Lemma 2.2 what we must prove is $b_i(\Delta;\mathbb{Q}) = \mu_i(\Delta;\mathbb{Q})$ for $i = 1, 2, \ldots, d-1$.

First, we prove $b_i(\Delta;\mathbb{Q}) = \mu_i(\Delta;\mathbb{Q})$ when $i = r$ and $i = d - r$. By Theorem 2.1(iv) and the Poincaré duality, it is enough to consider the case when $i = r$. Let $V$ be the vertex set of $\Delta$. Since $\Delta$ is 2-neighborly, $f_0(\mathrm{lk}_\Delta(v)) = n - 1$ for any $v \in V$. Since Theorems 2.1(i) and 3.1 say

$$b_r(\Delta;\mathbb{Q}) \leq \mu_r(\Delta;\mathbb{Q}) = \sum_{v \in V} \frac{\sigma_{r-1}(\mathrm{lk}_\Delta(v);\mathbb{Q})}{n} \leq \frac{1}{\binom{d+2}{r+1}}\binom{n-d-2+r}{r+1}, \tag{5}$$

we have $b_r(\Delta;\mathbb{Q}) = \mu_r(\Delta;\mathbb{Q})$ by the assumption.

Second, we prove $b_i(\Delta;\mathbb{Q}) = \mu_i(\Delta;\mathbb{Q}) = 0$ for $i \notin \{0, r, d-r, d\}$. Since we have the equality in (5), for any vertex $v$ of $\Delta$, $\mathrm{lk}_\Delta(v)$ satisfies $\binom{d+2}{r+1}\sigma_{r-1}(\mathrm{lk}_\Delta(v);\mathbb{Q}) = \binom{n-d-2+r}{r+1}$. Then Corollary 3.8 proves $\mu_i(\Delta;\mathbb{Q}) = 0$ for all $i$ with $i \notin \{0, r, d-r, d\}$. This fact and Theorem 2.1(i) imply $b_i(\Delta;\mathbb{F}) = \mu_i(\Delta;\mathbb{Q}) = 0$ for $i \notin \{0, r, d-r, d\}$, as desired. ☐

## 4. Tight combinatorial manifolds having simple homology groups

In this section, we prove Theorem 1.2(ii). In the rest of this paper, we assume that $\mathbb{F}$ is an infinite field.

We first introduce some known results on graded Betti numbers. Let $S = \mathbb{F}[x_1, \dots, x_n]$. For a homogeneous ideal $I \subset S$, let $I_{\leq k}$ be the ideal of $S$ generated by all polynomials in $I$ of degree $\leq k$. The following property is known (see [HH, Lemma 8.2.12]).

**Lemma 4.1.** *Let $I \subset S$ be a homogenous ideal. Then $\beta^S_{i,i+k}(S/I) = \beta^S_{i,i+k}(S/I_{\leq j})$ for all $i \geq 0$ and $k < j$.*

The next property proved by Fernández-Ramos and Gimenez [FG, Corollary 2.1] (for $r = 2$) and by Herzog and Srinivasan [HS, Corollary 4] (for general $r$) is known as (a special case of) the *subadditivity condition* of syzygies. See [ACI] for more information on the subadditivity condition.

**Theorem 4.2** (Herzog–Srinivasan)**.** *Let $I$ be an ideal generated by monomials of degree $\leq r$. Then*

$$\max\{k \in \mathbb{Z} : \beta^S_{i+1,k}(S/I) \neq 0\} \leq \max\{k \in \mathbb{Z} : \beta^S_{i,k}(S/I) \neq 0\} + r \quad \textit{for all } i \geq 0.$$

We say that a homogeneous ideal $I \subset S$ has an *$r$-linear resolution* if $\beta^S_{i,i+j}(I) = 0$ for all $i \geq 0$ and $j \neq r$. Since $\beta^S_{0,j}(I)$ is the number of degree $j$ elements in a minimal generating set of $I$, if $I$ has an $r$-linear resolution then $I$ is generated by degree $r$ polynomials. The next statement is an immediate consequence of Theorem 4.2.

**Corollary 4.3.** *Let $I \subset S$ be an ideal generated by monomials of degree $r$. If $\beta^S_{i,i+j}(I) = 0$ for all $i \geq 0$ and $j = r+1, r+2, \dots, 2r-1$, then $I$ has an $r$-linear resolution.*

Let $I \subset S$ be a homogeneous ideal. The *Krull dimension* $\dim S/I$ of $S/I$ is the minimal integer $k$ such that there is a sequence $\theta_1, \dots, \theta_k \in S$ of linear forms such that

$$\dim_{\mathbb{F}} S/(I + (\theta_1, \dots, \theta_k)) < \infty.$$

If $\dim S/I = d$, then a sequence $\Theta = \theta_1, \dots, \theta_d$ satisfying $\dim_{\mathbb{F}} S/(I + (\Theta)) < \infty$ is called a *linear system of parameters* (l.s.o.p. for short) of $S/I$. The ring $S/I$ is said to be *Cohen–Macaulay* if, for any l.s.o.p. $\Theta = \theta_1, \dots, \theta_d$ of $S/I$, where $d = \dim S/I$, $\theta_i$ is a non-zero divisor of $S/(I + (\theta_1, \dots, \theta_{i-1}))$ for $i = 1, 2, \dots, d$. Note that the Krull dimension of the Stanley–Reisner ring $\mathbb{F}[\Delta]$ is the dimension of $\Delta$ plus one [St, II Theorem 1.3].

The following fact, which essentially appears in [BH, Exercise 4.1.17], gives a connection between linear resolutions and the results in the previous section.

**Lemma 4.4.** *Let $I \subset S$ be a homogeneous ideal such that $S/I$ is Cohen–Macaulay of Krull dimension $d$ and let $R = \mathbb{F}[x_1, \dots, x_{n-d}]$. If $I$ has an $r$-linear resolution then $\beta^S_{i,j}(S/I) = \beta^R_{i,j}(R/(x_1, \dots, x_{n-d})^r)$ for all $i, j$.*

*Proof.* Let $\Theta = \theta_1, \dots, \theta_d$ be an l.s.o.p. of $S/I$. Then, since $\theta_k$ is a non-zero divisor of $S/(I + (\theta_1, \dots, \theta_{k-1}))$ for all $k$,

$$\beta^S_{i,j}(S/I) = \beta^{S/(\Theta)}_{i,j}(S/(I + (\Theta))) \tag{6}$$

for all $i, j$ (see e.g. [BH, Proposition 1.1.5] and [HH, Proposition A.2.2]). Observe that $S/(\Theta) \cong R$ as a ring. By this isomorphism, there is a homogeneous ideal $J \subset R$ such that $S/(I + (\Theta)) \cong R/J$ and

$$\beta^{S/(\Theta)}_{i,j}(S/(I + (\Theta))) = \beta^R_{i,j}(R/J)$$

for all $i, j$. We claim $J = (x_1, \dots, x_{n-d})^r$.

Since
$$\mathrm{Tor}^R_{n-d}(R/J, \mathbb{F})_{n-d+j} \cong \{f \in (R/J)_j : (x_1, \dots, x_{n-d})f = 0\}$$
for all $j \geq 0$ (see [HH, p. 268 and Corollary A.3.5]) and since $R/J$ is a finite dimensional $\mathbb{F}$-vector space, we have
$$\begin{aligned}\max\{j : (R/J)_j \neq 0\} &= \max\{j : \beta^R_{n-d,n-d+j}(R/J) \neq 0\} \\ &= \max\{j : \beta^S_{n-d,n-d+j}(S/I) \neq 0\} = r - 1,\end{aligned}$$
where the last equality follows from the assumption that $I$ has an $r$-linear resolution. The above equation implies $J \supset (x_1, \dots, x_{n-d})^r$. Also, since $I$ is generated by polynomials of degree $r$, $J$ is also generated by polynomials of degree $r$. These facts prove $J = (x_1, \dots, x_{n-d})^r$. □

For a homogeneous ideal $I \subset S$, the integer
$$\mathrm{depth}(S/I) = n - \max\{i : \beta^S_{i,j}(S/I) \neq 0 \text{ for some } j\}$$
is called the *depth* of $S/I$. (This is not a usual definition of the depth, but is equivalent to the usual one by the Auslander–Buchsbaum formula [BH, Theorem 1.3.3].) The following fact is fundamental in commutative algebra. See [BH, Proposition 1.2.12 and Section 2.1].

**Lemma 4.5.** *Let $I \subset S$ be a homogeneous ideal. Then* $\mathrm{depth}(S/I) \leq \dim(S/I)$, *and the equality holds if and only if $S/I$ is Cohen–Macaulay.*

The following symmetry of graded Betti numbers immediately follows from the Alexander duality [Spa, p. 296] and the Hochster's formula.

**Lemma 4.6.** *Let $\Delta$ be a triangulation of a $(d-1)$-sphere with the vertex set $[n]$. Then, for any $W \subset [n]$, $\widetilde{b}_{j-1}(\Delta_W; \mathbb{F}) = \widetilde{b}_{d-1-j}(\Delta_{[n]\setminus W}; \mathbb{F})$ for all $j$. In particular, we have $\beta_{i,i+j}(\mathbb{F}[\Delta]) = \beta_{n-d-i,n-i-j}(\mathbb{F}[\Delta])$ for all $i, j$.*

We now verify the following result which will serve as the key lemma in the proof of Theorem 1.2(ii). For a simplicial complex $\Delta$, its *$k$-skeleton* is the simplicial complex $\{F \in \Delta : \#F \leq k + 1\}$.

**Lemma 4.7.** *Let $\Delta$ be a triangulation of a $(d-1)$-sphere on $[n]$, $r \leq \frac{d-1}{3}$ a positive integer and $R = \mathbb{F}[x_1, \dots, x_{n-d-1}]$. If $\beta_{i,i+j}(\mathbb{F}[\Delta]) = 0$ for all $i \geq 0$ and $j \notin \{0, r, d-r, d\}$, then $\beta_{i,i+r}(\mathbb{F}[\Delta]) = \binom{i-1+r}{r}\binom{n-d-1+r}{i+r}$ for all $i \geq 0$.*

*Proof.* Since $\beta^S_{0,j}(I_\Delta) = \beta_{1,j}(\mathbb{F}[\Delta]) = 0$ for $r + 1 < j \leq d - r$ by the assumption, $(I_\Delta)_{\leq r+1} = (I_\Delta)_{\leq d-r}$. Thus by Lemma 4.1
$$\beta^S_{i,i+j}(S/(I_\Delta)_{\leq r+1}) = \beta^S_{i,i+j}(S/I_\Delta) \tag{7}$$
for all $i$ and $j \leq d - r - 1$. Then by Lemmas 3.4(i) and 4.4 it is enough to prove that $(I_\Delta)_{\leq r+1}$ has an $(r + 1)$-linear resolution and $S/(I_\Delta)_{\leq r+1}$ is Cohen–Macaulay of Krull dimension $d + 1$.

The assumption says $\beta_{0,j}(I_\Delta) = \beta_{1,j}(\mathbb{F}[\Delta]) = 0$ for $j \leq r$. Thus $(I_\Delta)_{\leq r+1}$ is generated by monomials of degree $r + 1$. Observe $d - r \geq 2r + 1$. Since (7) says
$$\beta^S_{i,i+j}((I_\Delta)_{\leq r+1}) = \beta^S_{i+1,i+j}(S/I_\Delta) = 0$$

for all $i \geq 0$ and $j = r+2, r+3, \ldots, d-r$, $(I_\Delta)_{\leq r+1}$ has an $(r+1)$-linear resolution by Corollary 4.3. It remains to prove that $S/(I_\Delta)_{\leq r+1}$ is Cohen–Macaulay of Krull dimension $d+1$. By (7) and Lemma 4.6,

$$\beta^S_{i,i+r}(S/(I_\Delta)_{\leq r+1}) = \beta^S_{i,i+r}(S/I_\Delta) = \beta^S_{n-d-i,n-i-r}(S/I_\Delta) = 0$$

for $i \geq n-d$. Then since $(I_\Delta)_{\leq r+1}$ has an $(r+1)$-linear resolution,

$$\text{depth}(S/(I_\Delta)_{\leq r+1}) = n - \max\{i : \beta^S_{i,i+r}(S/(I_\Delta)_{\leq r+1}) \neq 0\} \geq d+1. \tag{8}$$

Let $\Sigma$ be the simplicial complex defined by the equation $I_\Sigma = (I_\Delta)_{\leq r+1} = (I_\Delta)_{\leq d-r}$. Then

$$\Sigma = \{F \subset [n] : \Delta \text{ contains all subsets of } F \text{ of cardinality} \leq d-r\}.$$

We claim that $\Sigma$ has dimension $\leq d$.

Since $r \leq \frac{d-1}{2}$, if $\Sigma$ has dimension $\geq d+1$, then $\Delta$ contains the $\lfloor \frac{d-1}{2} \rfloor$-skeleton of a $(d+1)$-simplex. However, since the van Kampen–Flores theorem [Fl, vK] says that the $k$-skeleton $\Gamma_k$ of a $(2k+2)$-dimensional simplex cannot be embedded into $\mathbb{S}^{2k}$ and the cone of $\Gamma_k$ cannot be embedded into $\mathbb{S}^{2k+1}$ for any integer $k \geq 1$, $\Delta$ cannot contain the $\lfloor \frac{d-1}{2} \rfloor$-skeleton of a $(d+1)$-simplex. Thus $\Sigma$ has dimension at most $d$, and therefore $\dim S/(I_\Delta)_{\leq r+1} = \dim S/I_\Sigma \leq d+1$. By (8) and Lemma 4.5, $S/(I_\Delta)_{\leq r+1}$ is Cohen–Macaulay of Krull dimension $d+1$, as desired. □

**Remark 4.8.** The proof of Lemma 4.7 says that $\Delta$ is $r$-stacked, that is, it is the boundary of a homology $d$-ball all whose interior faces have dimension $\geq d-r$. This fact follows from the proof of [MuN1, Theorem 5.3] by using the fact that $S/(I_\Delta)_{\leq r+1}$ is Cohen–Macaulay of Krull dimension $d+1$ and that $\text{Tor}^S_i(S/(I_\Delta)_{\leq r+1}, \mathbb{F})_{i+j} = 0$ for all $i \geq 0$ and $j \geq r+1$.

We now prove Theorem 1.2(ii).

*Proof of Theorem 1.2(ii).* Since $\Delta$ is $\mathbb{F}$-tight, $\mu_i(\Delta;\mathbb{F}) = b_i(\Delta;\mathbb{F}) = 0$ for all integers $i$ with $i \notin \{0, r, d-r, d\}$. This implies that for every vertex $v$ of $\Delta$, $\sigma_{i-1}(\text{lk}_\Delta(v);\mathbb{F}) = 0$ if $i \notin \{0, r, d-r, d\}$. Then, by (3), each vertex link of $\Delta$ satisfies the assumption of Lemma 4.7. Since $\Delta$ is 2-neighborly, every vertex link of $\Delta$ has $n-1$ vertices. Thus, for each vertex $v$ of $\Delta$, $\beta_{i,i+r}(\mathbb{F}[\text{lk}_\Delta(v)]) = \binom{i-1+r}{r}\binom{n-d-2+r}{i+r}$ by Lemma 4.7. Then by Lemma 3.6 we have

$$\sigma_{r-1}(\text{lk}_\Delta(v);\mathbb{F}) = \sum_{k=r}^{n-1} \frac{\beta_{k-r,(k-r)+r}(\mathbb{F}[\text{lk}_\Delta(v)])}{\binom{n-1}{k}} = \frac{1}{\binom{d+2}{r+1}}\binom{n-d-2+r}{r+1}$$

for any vertex $v$ of $\Delta$. Since $\mathbb{F}$-tightness implies $b_r(\Delta;\mathbb{F}) = \mu_r(\Delta;\mathbb{F})$,

$$b_r(\Delta;\mathbb{F}) = \mu_r(\Delta;\mathbb{F}) = \sum_{v:\ \text{vertex of } \Delta} \frac{\sigma_{r-1}(\text{lk}_\Delta(v);\mathbb{F})}{n} = \frac{1}{\binom{d+2}{r+1}}\binom{n-d-2+r}{r+1},$$

as desired. □

## 5. Lower bounds on the numbers of vertices and edges

In this section, we discuss connections between Conjecture 1.4 and graded Betti numbers of Stanley–Reisner rings.

We consider classes of simplicial complexes which are more general than the class of combinatorial manifolds. A simplicial complex $\Delta$ of dimension $d$ is called an $\mathbb{F}$-*homology d-sphere* (or a *Gorenstein* complex* over $\mathbb{F}$ in some literatures) if, for any $F \in \Delta$ (including the empty face), $\widetilde{H}_{d-\#F}(\mathrm{lk}_\Delta(F);\mathbb{F}) \cong \mathbb{F}$ and $\widetilde{H}_k(\mathrm{lk}_\Delta(F);\mathbb{F}) = 0$ for $k \neq d-\#F$. An $\mathbb{F}$-*homology d-manifold* is a simplicial complex all whose vertex links are $\mathbb{F}$-homology $(d-1)$-spheres. A *normal pseudomanifold* of dimension $d$ is a $d$-dimensional connected pure simplicial complex satisfying that (i) every $(d-1)$-face is contained in exactly two facets, and (ii) the link of every face of dimension $\leq d-2$ is connected. Note that a combinatorial $d$-manifold is an $\mathbb{F}$-homology $d$-manifold for any field $\mathbb{F}$, and a connected $\mathbb{F}$-homology manifold is a normal pseudomanifold. A *stacked simplicial d-manifold* (resp. *d-sphere*) is the boundary of a triangulation of a $(d+1)$-manifold (resp. $(d+1)$-ball) all whose interior faces have dimension $\geq d$. When $d \geq 4$, a simplicial complex is a stacked simplicial $d$-manifold if and only if all its vertex links are stacked simplicial $(d-1)$-spheres [MuN2, Theorem 4.6]. Note that stacked simplicial spheres are exactly the boundaries of stacked polytopes.

The main results of this section are the following results which give lower bounds on the numbers of vertices and edges of homology manifolds and normal pseudomanifolds.

**Theorem 5.1.** *If $\Delta$ is a connected locally polytopal combinatorial d-manifold with n vertices, then $\binom{n-d-2+r}{r+1} \geq \binom{d+2}{r+1} b_r(\Delta;\mathbb{Q})$ for $r < \frac{d}{2}$.*

**Theorem 5.2.** *If $\Delta$ is a connected $\mathbb{F}$-homology $2r$-manifold with n vertices, then*

$$\binom{n-r-2}{r+1} \geq \binom{2r+2}{r+1} \frac{b_r(\Delta;\mathbb{F})}{2}.$$

**Theorem 5.3.** *Let $\Delta$ be a normal pseudomanifold of dimension $d \geq 3$ with $n$ vertices. Then*

(i) $f_1(\Delta) \geq (d+1)n + \binom{d+2}{2}(b_1(\Delta;\mathbb{F}) - 1)$. *The equality holds if and only if $\Delta$ is a stacked simplicial d-manifold.*
(ii) $\binom{n-d-1}{2} \geq \binom{d+2}{2} b_1(\Delta;\mathbb{F})$.

The above three results were proved by Novik and Swartz [NS1, NS2] for $\mathbb{F}$-orientable homology manifolds except for the equality case of Theorem 5.3 when $d = 3$. They also proved the inequalities in Theorem 5.3 for non-orientable 3-manifolds in [NS3, Theorem 4.9] (see [Sw, Remark 2.8]), and Bagchi [Bag1, Theorem 1.14] proved that the equality case of Theorem 5.3 also holds for all homology 3-manifolds. The above theorems extend their results to non-orientable homology manifolds of any dimension.

We prove the above theorems in the rest of this section. The main idea of the proof is to find upper bounds on $\sigma$-numbers which imply the desired inequalities by giving upper bounds on graded Betti numbers. We need two known results. The next result appears in [MiN, Corollary 8.5].

**Lemma 5.4.** *Let $S = \mathbb{F}[x_1, \dots, x_n]$, $I \subset S$ a homogeneous ideal and $w \in S$ a linear form. For a positive integer $j$, if the multiplication map*

$$\times w : (S/I)_k \to (S/I)_{k+1}$$

*is injective for all $k \le j$, then*

$$\beta^S_{i,i+k}(S/I) \le \beta^{S/wS}_{i,i+k}\big(S/(I + (w))\big).$$

*for all $i \ge 0$ and $k \le j$.*

The next result was proved in Fogelsanger's thesis [Fo] on the generic rigidity. We use an algebraic interpretation of his result given in [NS2, Section 5].

**Lemma 5.5** (Fogelsanger)**.** *Let $\Delta$ be a normal pseudomanifold of dimension $d-1 \ge 2$. There are linear forms $\theta_1, \dots, \theta_{d+1}$ such that the multiplication map*

$$\times\theta_i : \big(\mathbb{F}[\Delta]/(\theta_1, \dots, \theta_{i-1})\mathbb{F}[\Delta]\big)_k \to \big(\mathbb{F}[\Delta]/(\theta_1, \dots, \theta_{i-1})\mathbb{F}[\Delta]\big)_{k+1}$$

*is injective for all $i = 1, 2, \dots, d+1$ and $k \le 1$.*

We say that a simplicial complex $\Delta$ is *Cohen–Macaulay* (over $\mathbb{F}$) if $\mathbb{F}[\Delta]$ is a Cohen–Macaulay ring. Lemmas 3.4, 5.4 and 5.5 imply the following statement.

**Lemma 5.6.** *Let $\Delta$ be a $(d-1)$-dimensional simplicial complex with $n$ vertices.*

(i) *If $\Delta$ is Cohen–Macaulay over $\mathbb{F}$, then, for a positive integer $j$,*

$$\beta_{i,i+j}(\mathbb{F}[\Delta]) \le \binom{i-1+j}{j}\binom{n-d+j}{i+j}$$

*for all $i$. If the equality holds for all $i$, then $I_\Delta$ has a $(j+1)$-linear resolution.*

(ii) *If $\Delta$ is a normal pseudomanifold and $d \ge 3$, then*

$$\beta_{i,i+1}(\mathbb{F}[\Delta]) \le \binom{i}{1}\binom{n-d}{i+1} \quad \text{for all } i \ge 0.$$

*Proof.* (i) Suppose that $\Delta$ is a Cohen–Macaulay complex with the vertex set $[n]$. Let $S = \mathbb{F}[x_1, \dots, x_n]$ and $R = \mathbb{F}[x_1, \dots, x_{n-d}]$. Let $\Theta = \theta_1, \dots, \theta_d$ be an l.s.o.p. of $\mathbb{F}[\Delta]$. Since each $\theta_i$ is a non-zero divisor of $S/(I_\Delta + (\theta_1, \dots, \theta_{i-1}))$, by (6) and Lemma 3.4(ii)

$$\beta_{i,i+j}(\mathbb{F}[\Delta]) = \beta^{S/\Theta S}_{i,i+j}(S/(I_\Delta + (\Theta))) \le \beta^R_{i,i+j}(R/(x_1, \dots, x_{n-d})^{j+1})$$

for all $i$ and $j$. Then the inequality follows from Lemma 3.4(i). Also, if the equality holds, then $S/(I_\Delta + (\Theta)) \cong R/(x_1, \dots, x_{n-d})^{j+1}$ by Lemma 3.4(ii), which implies that $I_\Delta$ has a $(j+1)$-linear resolution since $(x_1, \dots, x_{n-d})^{j+1}$ has a $(j+1)$-linear resolution and since $\beta_{i,j}(\mathbb{F}[\Delta]) = \beta^{S/\Theta S}_{i,j}(S/(I_\Delta + (\Theta)))$ for all $i, j$.

(ii) Suppose that $\Delta$ is a normal pseudomanifold of dimension $d-1 \ge 2$ on $[n]$. Let $\Theta = \theta_1, \dots, \theta_{d+1}$ be linear forms given in Lemma 5.5 and let $R' = \mathbb{F}[x_1, \dots, x_{n-d-1}]$. Then in the same way as in the proof of (i) we have

$$\beta_{i,i+1}(\mathbb{F}[\Delta]) \le \beta^{S/\Theta S}_{i,i+1}(S/(I_\Delta + (\Theta))) \le \beta^{R'}_{i,i+1}(R'/(x_1, \dots, x_{n-d-1})^2)$$

by Lemma 5.4, and the assertion follows from Lemma 3.4(i). □

**Lemma 5.7.** *Let $\Delta$ be an $\mathbb{F}$-homology $(2r-1)$-sphere with $n$ vertices. Then*

$$\beta_{i,i+r}(\mathbb{F}[\Delta]) \leq \binom{i-1+r}{r}\binom{n-r-1}{i+r} + \binom{n-i-r-1}{r}\binom{n-r-1}{n-i-r}$$

*for all $i \geq 0$. If the equality holds for all $i$, then $\Delta$ is $r$-neighborly.*

*Proof.* Let $v$ be a vertex of $\Delta$. Consider the simplicial complex

$$\Gamma = \{F \in \Delta : v \notin F\} \cup \{F \cup \{v\} : F \in \Delta, v \notin F\}.$$

Thus $\Gamma$ is obtained from $\Delta$ by deleting the vertex $v$ and then taking a cone over $v$. By construction, $\Gamma$ is an $\mathbb{F}$-homology $2r$-ball whose boundary is $\Delta$, that is, $\Gamma$ is a Cohen–Macaulay simplicial complex of dimension $2r$ satisfying

- for each $F \in \Gamma$, $\widetilde{H}_{2r-\#F}(\mathrm{lk}_\Gamma(F);\mathbb{F})$ is either $\mathbb{F}$ or zero;
- $\{F \in \Gamma : \widetilde{H}_{2r-\#F}(\mathrm{lk}_\Gamma(F);\mathbb{F}) = 0\} = \Delta$.

Then it follows from [St, II Theorem 7.3] that $I_\Delta/I_\Gamma$ is the canonical module of $\mathbb{F}[\Gamma]$. Thus we have

$$\beta_{i,i+j}(I_\Delta/I_\Gamma) = \beta_{n-2r-1-i,n-i-j}(\mathbb{F}[\Gamma])$$

for all $i, j \geq 0$ (see [St, I Section 12]). Then, by the long exact sequence of Tor induced from the short exact sequence

$$0 \longrightarrow I_\Delta/I_\Gamma \longrightarrow \mathbb{F}[\Gamma] \longrightarrow \mathbb{F}[\Delta] \longrightarrow 0,$$

it follows that

$$\beta_{i,i+j}(\mathbb{F}[\Delta]) \leq \beta_{i,i+j}(\mathbb{F}[\Gamma]) + \beta_{i-1,i+j}(I_\Delta/I_\Gamma) = \beta_{i,i+j}(\mathbb{F}[\Gamma]) + \beta_{n-2r-i,n-i-j}(\mathbb{F}[\Gamma]) \tag{9}$$

for all $i, j$. By substituting the inequalities in Lemma 5.6(i) into the $j = r$ case of (9),

$$\beta_{i,i+r}(\mathbb{F}[\Delta]) \leq \binom{i-1+r}{r}\binom{n-r-1}{i+r} + \binom{n-i-r-1}{r}\binom{n-r-1}{n-i-r}$$

for all $i \geq 0$, as desired.

Suppose that the equality holds in the above inequality for all $i$. Then $\beta_{i,i+r}(\mathbb{F}[\Gamma]) = \binom{i-1+r}{r}\binom{n-r-1}{i+r}$ for all $i$, and therefore $\beta_{i,i+j}(\mathbb{F}[\Gamma]) = 0$ for all $i > 0$ and $j \neq r$ since $I_\Gamma$ has an $(r+1)$-linear resolution by Lemma 5.6(i). This fact and (9) say

$$\beta_{1,k}(\mathbb{F}[\Delta]) \leq \beta_{1,k}(\mathbb{F}[\Gamma]) + \beta_{n-2r-1,(n-2r-1)+(2r+1-k)}(\mathbb{F}[\Gamma]) = 0$$

for $k \neq r+1$. Thus $I_\Delta$ has no generators of degree $\leq r$, which implies the $r$-neighborliness of $\Delta$. □

Note that in Lemma 5.7 we assume that $\binom{a}{b} = 0$ if $a < b$.

**Corollary 5.8.** *Let $\Delta$ be a simplicial complex with $n$ vertices.*

(i) *If $\Delta$ is an $\mathbb{F}$-homology $(2r-1)$-sphere, then $\binom{2r+2}{r+1}\sigma_{r-1}(\Delta;\mathbb{F}) \leq 2\binom{n-r-1}{r+1}$. Moreover, if the equality holds, then $\Delta$ is $r$-neighborly.*

(ii) *If $\Delta$ is a normal pseudomanifold of dimension $d-1 \geq 2$ then $\binom{d+2}{2}\sigma_0(\Delta;\mathbb{F}) \leq \binom{n-d}{2}$. Moreover, the equality holds if and only if $\Delta$ is a stacked simplicial $(d-1)$-sphere.*

*Proof.* We first prove (i). By Lemma 5.7,

$$\begin{aligned}\sigma_{r-1}(\Delta;\mathbb{F}) &= \sum_{k=r}^{n} \frac{1}{\binom{n}{k}} \beta_{k-r,(k-r)+r}(\mathbb{F}[\Delta])\\ &\leq \sum_{k=0}^{n} \frac{1}{\binom{n}{k}} \binom{k-1}{r}\binom{n-r-1}{k} + \sum_{k=0}^{n} \frac{1}{\binom{n}{k}} \binom{n-k-1}{r}\binom{n-r-1}{n-k}\\ &= 2\left\{\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} \binom{k-1}{r}\binom{n-r-1}{k}\right\}\\ &= \frac{2}{\binom{2r+2}{r+1}}\binom{n-r-1}{r+1},\end{aligned}$$

as desired, where we use Lemma 3.6 for the last equality. The equality case also follows from Lemma 5.7.

We next prove (ii). In the same way as in the proof of (i), Lemmas 3.6 and 5.6(ii) imply the desired inequality

$$\sigma_0(\Delta;\mathbb{F}) \leq \sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \binom{k-1}{1}\binom{n-d}{k} = \frac{1}{\binom{d+2}{2}}\binom{n-d}{2}. \tag{10}$$

We prove that the equality holds in (10) if and only if $\Delta$ is a stacked simplicial $(d-1)$-sphere. Observe that the equality holds in (10) if and only if $\beta_{i,i+1}(\mathbb{F}[\Delta]) = \binom{i}{1}\binom{n-d}{i+1}$ for all $i$. It follows from [TH, Theorem 1.1] that if $\Delta$ is a stacked simplicial $(d-1)$-sphere then $\beta_{i,i+1}(\mathbb{F}[\Delta]) = \binom{i}{1}\binom{n-d}{i+1}$ for all $i$. Suppose that $\beta_{i,i+1}(\mathbb{F}[\Delta]) = \binom{i}{1}\binom{n-d}{i+1}$ for all $i$. We claim that $\Delta$ is a stacked simplicial $(d-1)$-sphere.

Since $\beta_{0,2}(I_\Delta) = \beta_{1,2}(\mathbb{F}[\Delta]) = \binom{n-d}{2}$, we have

$$f_1(\Delta) = \binom{n}{2} - \beta_{0,2}(I_\Delta) = \binom{n}{2} - \binom{n-d}{2} = dn - \binom{d+1}{2}.$$

If $d \geq 4$, then the above equation and the lower bound theorem (Theorem 1.5) prove that $\Delta$ is a stacked simplicial $(d-1)$-sphere. Suppose $d = 3$. Then $\Delta$ is a triangulation of a closed surface. Thus the Euler relation and the equation $2f_1(\Delta) = 3f_2(\Delta)$ imply $f_1(\Delta) = 3n - 3\chi(\Delta)$, where $\chi(\Delta)$ is the Euler characteristic of $\Delta$, and $\beta_{0,2}(I_\Delta) = \binom{n}{2} - (3n - 3\chi(\Delta))$. Since $\beta_{0,2}(I_\Delta) = \binom{n-3}{2} = \binom{n}{2} - 3n + 6$, we have $\chi(\Delta) = 2$, and therefore $\Delta$ is a triangulation of a 2-sphere. Then the desired statement follows from [BDSS, Theorem 1.1] which proved that a triangulation of a 2-sphere satisfies the equality in (10) if and only if it is a stacked simplicial 2-sphere. □

Now we prove Theorems 5.1, 5.2 and 5.3.

*Proof of Theorems 5.1 and 5.2.* Let $V$ be the vertex set of $\Delta$ and let $n_v = f_0(\mathrm{lk}_\Delta(v))$ for $v \in V$. We use the following inequality: For positive integers $a, b$ and $r$ with

$a+1 \geq b \geq 2$, one has

$$\frac{1}{a+1}\binom{(a+1)-b+r}{r} = \frac{a(a+1-b+r)}{(a+1)(a+1-b)}\left\{\frac{1}{a}\binom{a-b+r}{r}\right\} > \frac{1}{a}\binom{a-b+r}{r}, \tag{11}$$

where the inequality directly follows without passing the middle term when $b = a+1$.

Now we prove the statements. By Theorem 2.1(i)

$$\binom{d+2}{r+1} b_r(\Delta;\mathbb{Q}) \leq \binom{d+2}{r+1}\mu_r(\Delta;\mathbb{F}) = \sum_{v\in V}\binom{d+2}{r+1}\frac{\sigma_{r-1}(\mathrm{lk}_\Delta(v);\mathbb{F})}{n_v+1} \tag{12}$$

for all $r$. Suppose that $\Delta$ is locally polytopal. Then by Theorem 3.1 and (11)

$$\begin{aligned}\sum_{v\in V}\binom{d+2}{r+1}\frac{\sigma_{r-1}(\mathrm{lk}_\Delta(v);\mathbb{F})}{n_v+1} &\leq \sum_{v\in V}\frac{1}{n_v+1}\binom{(n_v+1)-(d+3)+r+1}{r+1}\\ &\leq \sum_{v\in V}\frac{1}{n}\binom{n-d-2+r}{r+1} = \binom{n-d-2+r}{r+1}\end{aligned}$$

for $r < \frac{d}{2}$, where we apply (11) when $a = n_v+1$ and $b = d+3$ for the second inequality. These inequalities prove Theorem 5.1. Similarly, Theorem 5.2 follows by substituting the inequality in Corollary 5.8(i) into the right-hand side of (12). □

*Proof of Theorem 5.3.* The statement (ii) follows from (i) by substituting inequality $f_1(\Delta) \leq \binom{n}{2}$ into the left-hand side of the inequality (i) (this fact was observed in [LSS, Theorem 5]). We prove (i). Since a link of a normal pseudomanifold is again a normal pseudomanifold, by Theorem 2.1(v) and Corollary 5.8(ii)

$$\begin{aligned}\binom{d+2}{2}(b_1(\Delta;\mathbb{F})-1) &\leq \binom{d+2}{2}(\mu_1(\Delta;\mathbb{F})-\mu_0(\Delta;\mathbb{F}))\\ &= \binom{d+2}{2}\sum_{v\in V}\frac{\sigma_0(\mathrm{lk}_\Delta(v))-1}{f_0(\mathrm{lk}_\Delta(v))+1}\\ &\leq \sum_{v\in V}\frac{1}{f_0(\mathrm{lk}_\Delta(v))+1}\left\{\binom{f_0(\mathrm{lk}_\Delta(v))-d}{2}-\binom{d+2}{2}\right\}\\ &= f_1(\Delta)-(d+1)f_0(\Delta),\end{aligned}$$

which implies the desired inequality. Also, when $d \geq 4$, since a simplicial complex is a stacked simplicial $d$-manifold if and only if all its vertex links are stacked simplicial $(d-1)$-spheres, it follows from Corollary 5.8(ii) that the equality holds in the above inequality if and only if $\Delta$ is a stacked simplicial $d$-manifold.

It remains to prove the equality case when $d = 3$. Since if the equality holds in the above inequality then every vertex link of $\Delta$ is a stacked simplicial sphere by Corollary 5.8(ii), we may assume that $\Delta$ is an $\mathbb{F}$-homology 3-manifold. Then the desired statement follows from [Bag1, Theorem 1.14]. □

**Corollary 5.9.** *Let $\Delta$ be a connected $\mathbb{F}$-homology $2r$-manifold with $n$ vertices. If*

$$\binom{n-r-2}{r+1} = \binom{2r+2}{r+1}\frac{b_r(\Delta;\mathbb{F})}{2},$$

*then* $\Delta$ *is* $(r+1)$*-neighborly. Moreover, if* $\Delta$ *is in addition* $\mathbb{F}$*-orientable, then* $\Delta$ *is* $\mathbb{F}$*-tight.*

*Proof.* Suppose $\binom{n-r-2}{r+1} = \binom{2r+2}{r+1}\frac{b_r(\Delta;\mathbb{F})}{2}$. Then the proof of Theorem 5.2 says that each vertex link of $\Delta$ has $n-1$ vertices and satisfies the equality in Corollary 5.8(i). Thus each vertex link of $\Delta$ is $r$-neighborly and has $n-1$ vertices, which implies that $\Delta$ is $(r+1)$-neighborly. Also, if $\Delta$ is in addition $\mathbb{F}$-orientable, then it is $\mathbb{F}$-tight by [Kü, Corollary 4.7] and [Bag2, Theorem 12]. □

Every stacked simplicial $d$-manifold is obtained from a stacked simplicial $d$-sphere by applying combinatorial handle additions repeatedly. See [DM, Ka]. If $\Delta$ is a stacked simplicial $d$-manifold with $n$ vertices, then its face numbers only depend on $n, d$ and $b_1(\Delta;\mathbb{F})$. They are given by

$$f_j(\Delta) = \begin{cases} \binom{d+1}{j}n + j\binom{d+2}{j+1}(b_1(\Delta;\mathbb{F})-1), & \text{if } 1 \le j < d, \\ dn + (d-1)(d+2)(b_1(\Delta;\mathbb{F})-1), & \text{if } j = d. \end{cases}$$

See [BD, Theorem 3.12]. It is known that Theorem 5.3 implies the following consequence on face numbers of normal pseudomanifolds. (We omit the proof since it is the same as the proof of [BD, Theorem 3.12].)

**Corollary 5.10.** *Let* $\Delta$ *be a normal pseudomanifold of dimension* $d \ge 3$ *with* $n$ *vertices. Then*

$$f_j(\Delta) \ge \begin{cases} \binom{d+1}{j}n + j\binom{d+2}{j+1}(b_1(\Delta;\mathbb{F})-1), & \text{if } 1 \le j < d, \\ dn + (d-1)(d+2)(b_1(\Delta;\mathbb{F})-1), & \text{if } j = d. \end{cases}$$

*The equality holds for some* $j \ge 1$ *if and only if* $\Delta$ *is a stacked simplicial* $d$*-manifold.*

## 6. Concluding remarks and Questions

In this section, we write some remarks and questions related to Conjectures 1.1 and 1.4.

**Necessity of Conjecture 1.1.** To approach the only if part of Conjecture 1.1, one may ask if it is possible to extend Theorem 1.2(ii) to all positive integers $r$ and $d$ with $r < \frac{d}{2}$. Unfortunately, this is not possible. Indeed, the 13 vertex $(\mathbb{Z}/2\mathbb{Z})$-tight triangulation $\Delta$ of the 5-dimensional manifold $SU(3)/SO(3)$ given in [KL, p. 170] satisfies $b_1(\Delta;\mathbb{Z}/2\mathbb{Z}) = b_4(\Delta;\mathbb{Z}/2\mathbb{Z}) = 0$ and $b_2(\Delta;\mathbb{Z}/2\mathbb{Z}) = 1$. However,

$$\binom{13-5-2+2}{2+1} = \binom{8}{3} > \binom{7}{3} = \binom{5+2}{2+1},$$

which means that $\Delta$ does not satisfy the equation in Theorem 1.2. On the other hand, we are not sure if the assumption $r \le \frac{d-1}{3}$ is sharp for the conclusion of Theorem 1.2(ii). On this problem, we ask

**Question 6.1.** Does the conclusion of Theorem 1.2(ii) hold under the weaker assumption $r < \frac{d-1}{2}$?

Even if Theorem 1.2(ii) is not extendable, the argument in Section 4 could be useful to study Conjecture 1.1. We pose the next conjecture which implies the only if part of Conjecture 1.1.

**Conjecture 6.2.** If $\Delta$ is an $\mathbb{F}$-tight combinatorial triangulation of $\mathbb{S}^i \times \mathbb{S}^j$ with $j > i$, then $(I_{\mathrm{lk}_\Delta(v)})_{\leq i+1}$ has an $(i+1)$-linear resolution for any vertex $v$ of $\Delta$.

Recently, Spreer [Spr, Theorem 1.1] gave an interesting upper bound on the number of vertices of $\mathbb{F}$-tight triangulations of $(\ell-1)$-connected closed $(2\ell+1)$-manifolds. This result gives upper bounds on the number of vertices of $\mathbb{F}$-tight triangulations of $\mathbb{S}^\ell \times \mathbb{S}^{\ell+1}$ which is close to $3\ell+6$ suggested in Conjecture 1.1. It would be interesting to study the conjecture in this case.

**Sufficiency of Conjecture 1.1.** In Theorem 1.2(i), we need the local polytopality assumption since we use the results of Migliore and Nagel [MiN]. Their results actually hold not only for polytopes but also for homology spheres having the weak Lefschetz property (see [MiN, MuN1, NS2] for the definition of the weak Lefschetz property). It was conjectured that every homology sphere has the weak Lefschetz property, but we re-ask the following special case of this conjecture since it will prove the if part of Conjecture 1.1.

**Problem 6.3.** Prove that every combinatorial $(d-1)$-sphere with at most $2d+1$ vertices has the weak Lefschetz property.

**Lower bounds on the number of vertices.** The proofs given in Section 5 say that upper bounds on graded Betti numbers of homology spheres induce lower bounds on the number of vertices of homology manifolds. In particular, the proof of Theorem 5.1 says that, to prove Conjecture 1.4, it is enough to prove the following upper bounds on graded Betti numbers.

**Conjecture 6.4.** Let $\Delta$ be an $\mathbb{F}$-homology $(d-1)$-sphere with $n$ vertices and let $R = \mathbb{F}[x_1, \dots, x_{n-d-1}]$. Then, for all $i \geq 0$ and $1 \leq r < \frac{d}{2}$, one has

$$\beta_{i,i+r}(\mathbb{F}[\Delta]) \leq \beta^R_{i,i+r}(R/(x_1, \dots, x_{n-d-1})^{r+1}) = \binom{i-1+r}{r}\binom{n-d-1+r}{i+r}.$$

Lemma 5.6 says that Conjecture 6.4 holds when $r = 1$. Also, the conjecture holds for simplicial polytopes when $\mathbb{F} = \mathbb{Q}$ by the result of Migliore and Nagel.

Corollary 5.10 gives lower bounds on face numbers of normal pseudomanifolds, and they are sharp if $n$ is sufficiently large. However, the following question is open.

**Question 6.5.** Fix positive integers $d \geq 3$ and $b$. What is the minimal number $n$ such that there is a stacked simplicial $d$-manifold $\Delta$ with $f_0(\Delta) = n$ and $b_1(\Delta; \mathbb{F}) = b$?

Theorem 5.3(ii) says that $n$ and $b$ must satisfy $\binom{n-d-1}{2} \geq \binom{d+2}{2} b$. This inequality is known to be sharp in some cases. For example, the equality holds when $b = 1$ and when $b = d^2 + 5d + 6$ [DS]. But we are not sure if this inequality is enough to answer the above question. Note that the first Betti number of a stacked simplicial $d$-manifold does not depend on the choice of the base field $\mathbb{F}$ when $d \geq 3$.

**Existence of tight triangulations.** There are infinite number of $\mathbb{F}$-tight triangulations satisfying the assumption of Theorem 1.2(ii) when $r = 1$. See [DS, Section 6]. On the other hand, it is quite hard to find tight triangulations of manifolds, and only finitely many examples of $\mathbb{F}$-tight combinatorial $d$-manifolds $\Delta$ with $b_2(\Delta; \mathbb{F}) \neq 0$ are known in dimension $\geq 3$. In particular, for products of two spheres, $\mathbb{S}^1 \times \mathbb{S}^{2k+1}$,

$\mathbb{S}^2 \times \mathbb{S}^3$ and $\mathbb{S}^3 \times \mathbb{S}^3$ seem to be the only cases when the existence of tight triangulations is known, where $k \in \mathbb{Z}_{\geq 0}$. See [KL]. In view of Corollary 1.3, the following problem might be tractable.

**Problem 6.6.** Study the existence of tight triangulations of $\mathbb{S}^i \times \mathbb{S}^j$ when $i \geq 2$ and $j > 2i$.

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