HILBERT SCHEMES and MAXIMAL BETTI NUMBERS
over
VERONESE RINGS

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Abstract: We show that Macaulay's Theorem, Gotzmann's Persistence Theorem, and Green's Theorem hold over a Veronese toric ring R. We also prove that the Hilbert scheme over R is connected; this is an analogue of Hartshorne's theorem that the Hilbert scheme over a polynomial ring is connected. Furthermore, we prove that each lex ideal in R has the greatest Betti numbers among all graded ideals with the same Hilbert function.

1. Introduction

Throughout this paper S stands for the polynomial ring $k[x_1, \ldots, x_n]$ over a field k of characteristic zero. The ring S is graded by $\deg(x_i) = 1$ for each i, and S_i stands for the vector space of all polynomials of degree i. Furthermore, if J is a graded ideal, then J_i stands for the vector space of all polynomials in J of degree i. The Hilbert function

$$\begin{aligned} h: \quad & \mathbf{N} \longrightarrow \mathbf{N} \\ & i \mapsto \dim_k J_i \end{aligned}$$

is an important numerical invariant. A key role in the study of Hilbert functions and syzygies is played by lex ideals. Lex ideals are special monomial ideals, defined in a simple combinatorial way. Macaulay's Theorem characterizes the Hilbert functions of graded ideals in S:

Macaulay's Theorem 1.1. [Ma] *For every graded ideal in S there exists a lex ideal with the same Hilbert function.*

The structure of the Hilbert scheme is usually very complicated. The main known structural result is Hartshorne's famous Theorem 1.2.

Theorem 1.2. [Ha] *The Hilbert scheme $\mathcal{H}_S^h$, that parametrizes all graded ideals with a fixed Hilbert function h, is connected. Every graded ideal in S with Hilbert function h is connected by a sequence of deformations to the lex ideal with Hilbert function h.*

As a consequence, Macaulay's Theorem was generalized to Betti numbers by Bigatti, Hulett, Pardue.

Theorem 1.3. [Bi, Hu, Pa] *Every lex ideal in S attains maximal Betti numbers among all graded ideals with the same Hilbert function.*

Analogues of these results are proved over an exterior algebra. The following results are known.

Theorem 1.4. *Let E be an exterior algebra on variables $e_1, \dots, e_n$ over k. The ring E is graded by $\deg(e_i) = 1$ for each i.*

(1) (Kruskal-Katona) [Kr, Ka] *For every graded ideal in E there exists a lex ideal with the same Hilbert function.*
(2) (Peeva-Stillman) [PS1] *The Hilbert scheme $\mathcal{H}_E^h$, that parametrizes all graded ideals with a fixed Hilbert function h, is connected. Every graded ideal in E with Hilbert function h is connected by a sequence of deformations to the lex ideal with Hilbert function h.*
(3) (Aramova-Herzog-Hibi) [AHH] *Every lex ideal in E attains maximal Betti numbers among all graded ideals with the same Hilbert function.*

(1), (2), and (3) are analogues to Theorems 1.1, 1.2, and 1.3 respectively. Furthermore, analogues of these results are proved over Clements-Lindström rings.

Theorem 1.5. *Consider a Clements-Lindström ring $C = k[x_1, \dots, x_n]/(x_1^{c_1}, \dots, x_n^{c_n})$, where $c_1 \leq \dots \leq c_n \leq \infty$.*

(1) (Clements-Lindström) [CL] *For every graded ideal in C there exists a lex ideal with the same Hilbert function.*

(2) (Murai-Peeva) [MP] *The Hilbert scheme $\mathcal{H}^h_C$, that parametrizes all graded ideals with a fixed Hilbert function h, is connected. Every graded ideal in C with Hilbert function h is connected by a sequence of deformations to the lex ideal with Hilbert function h.*

(3) (Murai-Peeva) [MP] *Every lex ideal in C attains maximal Betti numbers among all graded ideals with the same Hilbert function.*

The notion of a lex ideal over a projective toric ring is introduced in [GHP]. Our main results are analogues of the above results over Veronese rings; such rings have received a lot of interest in Commutative Algebra and Algebraic Geometry. We prove

Theorem 1.6. *Let I be the defining ideal of a Veronese toric ring (see Section 3 for a precise definition), and $R = S/I$.*

(1) For every graded ideal in R there exists a lex ideal with the same Hilbert function.

(2) The Hilbert scheme $\mathcal{H}^h_R$, that parametrizes all graded ideals with a fixed Hilbert function h, is connected. Every graded ideal in R with Hilbert function h is connected by a sequence of deformations to the lex ideal with Hilbert function h.

(3) Every lex ideal in R attains maximal Betti numbers among all graded ideals with the same Hilbert function.

The specific lex order, that we use in Theorem 1.6, is defined in Section 3.

Macaulay's Theorem 1.1 and Kruskal-Katona's Theorem 1.4.(1) yield a numerical characterization of the Hilbert functions over a polynomial ring and over an exterior algebra, respectively. Similarly, Theorem 1.6.(1) leads to a numerical characterization of the Hilbert functions over a Veronese ring; we present this in Section 4.

Gotzmann's Persistence Theorem [Go] is an important result on Hilbert functions. It is proved over polynomial rings in [Go] and over exterior algebras in [AHH]. It is an open question whether Gotzmann's Persistence Theorem holds over a Clements-Lindström ring. In Section 4, we prove that Gotzmann's Persistence Theorem holds over a Veronese ring. Another important result on Hilbert functions is Green's Theorem [Gr]; in 3.8 we show that it holds over R.

In Section 5 we discuss a more precise (but more technical) version of some of the results.

We raise the open problem whether the lex point on the Hilbert scheme is a smooth point. Reeves and Stillman [RS] proved that the lexicographic point is smooth on the classical Hilbert scheme, introduced by Grothendieck [Gr], that parametrizes subschemes of $\mathbf{P}^r$ with fixed Hilbert polynomial. It is an open problem whether the lexicographic point is smooth on the Hilbert scheme $\mathcal{H}_V^h$, that parametrizes all graded ideals with a fixed Hilbert function h in the ring V, where V is either a polynomial ring S, or an exterior algebra E, or a Clements-Lindstöm ring C, or a Veronese ring R. It is possible that the answer is different over different rings. The proof in [RS] does not work because the tangent space is different, see [PS2].

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2. Lex ideals

In this section, we construct special monomial orders and examples that illustrate the subtleness of the lex definition in a toric ring.

Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be a subset of $\mathbf{N}^r \setminus \{\mathbf{0}\}$, A be the matrix with columns $\mathbf{a}_i$, and suppose that $\text{rank}(A) = r$. For $1 \leq i \leq n$, denote $\mathbf{t}^{\mathbf{a}_i} = t_1^{a_{i1}} \ldots t_r^{a_{ir}}$, where $\mathbf{a}_i = (a_{i1}, \ldots, a_{ir})$. We denote by $I_\mathcal{A}$ the kernel of the homomorphism

$$\varphi_\mathcal{A} : \ k[x_1, \ldots, x_n] \to k[t_1, \ldots, t_r]$$
$$x_i \mapsto \mathbf{t}^{\mathbf{a}_i}\ .$$

It is a prime ideal, and is called the *toric ideal* associated to $\mathcal{A}$. The *toric ring* associated to $\mathcal{A}$ is $R_\mathcal{A} = S/I_\mathcal{A}$. We have the isomorphism $Q_\mathcal{A} := k[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}] \cong R_\mathcal{A}$. The toric ideal $I_\mathcal{A}$ is *projective* (or $S/I_\mathcal{A}$ is a *projective* toric ring) if $I_\mathcal{A}$ is homogeneous with respect to the standard grading of S with $\deg(x_i) = 1$ for $1 \leq i \leq n$. Then $R_\mathcal{A}$ inherits the grading from S. Let $p \geq 0$ be an integer; a *p-monomial space* W is a vector subspace of $(R_\mathcal{A})_p$ spanned by monomials of degree p. In the rest, we consider only projective toric rings.

Now, we focus on the definition of a lex ideal in a projective toric ring. For simplicity we will assume that the monomials $\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}$ have the same degree.

Construction 2.1. Order the variables in $T = k[t_1, \ldots, t_r]$ by $t_1 > t_2 > \ldots > t_r$ and consider the following monomial order $>$ in $Q_\mathcal{A}$: if v and v' are monomials in $Q_\mathcal{A}$, then $v > v'$ if v is degree-lex-greater than v'. This is a total order on the monomials in $Q_\mathcal{A}$ (and also in T). We denote it by $\mathbf{dlex}_T$.

Order the monomials in S so that $m >_T u$ in S if $\varphi_{\mathcal{A}}(m) >_{\mathbf{dlex}\,T} \varphi_{\mathcal{A}}(u)$ in T. This is a partial order on the monomials in S. Two monomials m and m' in S are incomparable by $>_T$ if and only if $\varphi_{\mathcal{A}}(m) = \varphi_{\mathcal{A}}(m')$, which holds if and only if $m - m' \in I_{\mathcal{A}}$.

Construction 2.2. We define a partial monomial order $<_{toric}$ on S using the weight orders with respect to the rows in the matrix A. For $1 \le i \le r$, denote by $\mathbf{w}_i$ the weight order of the monomials in S with respect to the vector $((\mathbf{a}_1)_i, \dots, (\mathbf{a}_n)_i)$. Let $\mathbf{w}_0 = \sum_{i=1}^{r} \mathbf{w}_i$. Let m and u be two monomials in S. We define that $m >_{toric} u$ if there exists a $0 \le j \le r$ such that

$$\mathbf{w}_j(m) > \mathbf{w}_j(u) \text{ and } \mathbf{w}_i(m) = \mathbf{w}_i(u) \text{ for } 1 \le i < j\,.$$

This is a partial order on the monomials in S.

Two monomials m and m' are incomparable by $<_{toric}$ if and only if $\mathbf{w}_i(m) = \mathbf{w}_i(m')$ for all $1 \le i \le r$; this happens if and only if $m - m' \in I_{\mathcal{A}}$. Hence, the following two properties hold:

(*a*) $\mathrm{in}_{<_{toric}}(I_{\mathcal{A}}) = I_{\mathcal{A}}$.

(*b*) if m and m' are incomparable monomials, then $m - m' \in I_{\mathcal{A}}$.

It is easy to check that the following lemma holds.

Lemma 2.3. *The two partial orders on S constructed in Constructions 2.1 and 2.2 coinside.*

Note that $<_T = <_{toric}$ induces a well-defined total monomial order $<_T$ in the ring $R_{\mathcal{A}}$ since $m - m' \in I_{\mathcal{A}}$ for two monomials m and m' in S if and only if the monomials are equal with respect to $<_T$.

Let M be a monomial ideal in $R_{\mathcal{A}}$. Denote by $\varphi_{\mathcal{A}}(M)$ its image ideal in $Q_{\mathcal{A}} = k[\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_n}]$. It looks natural to define that M is lex if $\varphi(M_q)$ is lex in $Q_{\mathcal{A}}$ for every $q \ge 0$. Example 2.4 shows that this is not a satisfactory definition since a lex monomial space (in a fixed degree) may not generate a lex monomial space in the next degree.

Example 2.4. Consider the defining ideal B of $\mathbf{P}^1 \times \mathbf{P}^1$. It is the kernel of the homomorphism

$$\mu: \ X = k[x_{11}, x_{12}, x_{21}, x_{22}] \to D = k[t_1, t_2, s_1, s_2]$$
$$x_{11} \mapsto t_1 s_1$$
$$x_{12} \mapsto t_1 s_2$$
$$x_{21} \mapsto t_2 s_1$$
$$x_{22} \mapsto t_2 s_2 \, .$$

Denote by $\mathbf{dlex}_D$ the degree-lex monomial order in the polynomial ring D so that the variables are ordered by $t_1 > t_2 > s_1 > s_2$. Furthermore, we order the variables in the polynomial ring X so that $x_{ij} > x_{pq}$ if $\mu(x_{ij}) >_{\mathbf{lex}_D} \mu(x_{pq})$. Therefore, $x_{11} > x_{12} > x_{21} > x_{22}$. Consider the degree-lex monomial order $\mathbf{dlex}_X$ on X determined by the equivalent Constructions 2.1 and 2.2.

Let $M = (x_{11})$ in the Segre toric ring X/B. On the one hand, $\mu(M_1)$ is spanned by $t_1 s_1$, so it is a lex monomial space in D.

On the other hand, $\mu(M_2)$ contains the monomial $\mu(x_{11}x_{21}) = t_1 t_2 s_1^2$, but does not contain the $\mathbf{dlex}_D$-bigger monomial $t_1^2 s_2^2 = \tau(x_{12}^2)$. Therefore, $\mu(M_2)$ is not a lex monomial space in D.

This example explains why a more intricate definition of lex is introduced in [GHP].

Definitions 2.5. [GHP] We fix the order of the variables in S to be $x_1 > \ldots > x_n$, and consider the induced dlex order $\mathbf{dlex}_S$ on S. The lex-greatest monomial in a fiber will be called the *top-representative* of the fiber.

We say that a p-monomial space W in $R_{\mathcal{A}}$ is a *lex space* if the following property is satisfied: if $m \in W$ is a monomial, $v \in S$ is the top-representative of the fiber of m, and $u \in S_p$ is a monomial such that $u >_{\mathbf{dlex}_S} v$, then $u \in W$ (by abuse of notation $q \in W$ means that the image of u in $R_{\mathcal{A}}$ is a monomial in W).

It is proved in [GHP] that a lex p-monomial space in $(R_{\mathcal{A}})_p$ generates a lex monomial space in $(R_{\mathcal{A}})_{p+1}$. A monomial ideal L in $R_{\mathcal{A}}$ is called *lex* if for every $i \geq 0$, we have that L_i is spanned by a lex monomial space in $(R_{\mathcal{A}})_i$.

In Example 2.4, it is easy to check that M_2 is lex monomial space according to the definition above. The fact that $\mu(x_{12}^2) = t_1^2 s_2^2 \notin \mu(M_2)$ is not a problem anymore since x_{12}^2 is not $\mathbf{dlex}_X$-bigger than any of the top-representatives of the fibers of the monomials in M_2.

We close this section by an example which shows that the $\mathbf{dlex}_D$ order in $Q_{\mathcal{A}}$ does not agree with the $\mathbf{dlex}_S$ order on the set of the top-representatives in S. This illustrates the subtleness of the definition of a lex ideal in a toric ring.

Example. 2.6. We continue Example 2.4. On the one hand, we have the inequality

$$x_{12}^2 <_{\mathbf{dlex}_X} x_{11}x_{21}\,.$$

On the other hand, we have that

$$\mu(x_{12}^2) = t_1^2 s_2^2 >_{\mathbf{dlex}_D} t_1 t_2 s_1^2 = \mu(x_{11}x_{21})\,.$$

Thus, the following property fails: $m >_{\mathbf{dlex}_X} u$ for two top-representative monomials in X if and only if $\mu(m) >_{\mathbf{dlex}_D} \mu(u)$. In this example, the $\mathbf{dlex}_D$ order in $Q_{\mathcal{A}}$ does not agree with the $\mathbf{dlex}_S$ order on the set of the top-representatives in S.

3. Veronese rings

After notation is introduced (say in some lemma or construction), it will be used in the rest of the paper. We will prove Theorem 1.6 in a series of constructions.

3.1. Veronese rings.

Fix integer numbers $r, q \geq 1$. Set $n = \binom{r+q-1}{r-1}$.

For an integer column vector $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, set $\mathbf{x}^{\mathbf{v}} = x_1^{v_1} \dots x_n^{v_n}$.

T stands for the polynomial ring $k[t_1, \dots, t_r]$, graded by $\deg(t_j) = 1$ for $1 \leq j \leq r$. Let R be the q'th Veronese ring in r variables which defines the q'th Veronese embedding of $\mathbf{P}^{r-1}$. Thus,

$$R \cong \oplus_{j=0}^{\infty} T_{jq} = k[\text{ all monomials of degree } q \text{ in } T\ ]\,.$$

The set of points defining the toric ideal can be taken to be

$$\mathcal{A} = \{(i_1, \dots, i_r) \in \mathbf{N}^r \setminus \mathbf{0} \,|\, \sum_{j=1}^{r} i_j = q\}\,.$$

We order the vectors in $\mathcal{A}$ lexicographically so that the first vector is the greatest, and then denote them by $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Let A be the matrix with columns $\mathbf{a}_i$. We denote $k[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}]$ by Q.

We denote by I the kernel of the homomorphism

$$\varphi: \; k[x_1, \ldots, x_n] \to k[t_1, \ldots, t_r]$$
$$x_i \mapsto \mathbf{t}^{\mathbf{a}_i} \,.$$

It is a prime ideal, and is called the *Veronese toric ideal* associated to $\mathcal{A}$. The *Veronese toric ring* associated to $\mathcal{A}$ is

$$R := S/I \cong k[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}] = Q \subseteq T \,.$$

Furthermore, φ_S stands for the map $S \to R$, and φ_R stands for the isomorphism $R \cong Q$ induced by φ. Thus, $\varphi = \varphi_R \varphi_S$.

The Veronese toric ideal I is projective, that is, I is homogeneous with respect to the standard grading of S with $\deg(x_i) = 1$ for $1 \leq i \leq n$. The ring R inherits the grading from S.

3.2. The lex property in Veronese rings.

In contrast to Example 2.6, it is easy to see that we have the following very useful lemma over Veronese rings. This lemma is the foundation for our arguments.

Lemma 3.2.1. *In the notation of Section 2, over a Veronese ring we have that the* $\mathbf{dlex}_T$ *order in* Q *agrees with the* $\mathbf{dlex}_S$ *order on the set of the top-representatives in* S.

For simplicity we call this order lex throughout.

An immediate consequence is the following result.

Theorem 3.2.2. *A monomial ideal* M *in the Veronese ring* R *is lex if and only if its image* $\varphi(M)$ *generates a lex ideal in the polynomial ring* T.

Construction 3.2.3. If G is an ideal in the polynomial ring T, then we set $G_Q = Q \cap G$ and $G_R = \varphi_R^{-1}(G_Q)$, and denote by G_S the preimage of G_R in S. Thus, G_R, and G_S are ideals in R and S, respectively. We also set $G_T = G$.

Construction 3.2.4. If N is an ideal in the Veronese ring R, then we denote by ${}_QN$ the isomorphic ideal $\varphi_R(N)$ in Q. Furthermore, denote by ${}_TN$ the ideal in T generated by ${}_QN$, and denote by ${}_SN$ the preimage of N in S. Thus, ${}_TN$ and ${}_SN$ are ideals in T, and S, respectively. We also set ${}_RN = N$.

Proposition 3.2.5.

(1) If G is a lex ideal in T, then G_R is lex in R, and G_S is lex+I in S.

(2) If N is a lex ideal in R, then ${}_TN$ is lex in T, and ${}_SN$ is lex+I in S.

We will consider three types of deformations, which we call Type-A deformations, Type-B deformations, and Type-C deformations.

3.3. Type-A deformations using change of coordinates.

The general linear group $\mathcal{G} = \mathrm{GL}(r,k)$ of invertible $(r \times r)$-matrices over k, acts as a group of algebra automorphisms on the polynomial ring T by acting on the variables as follows: if $E \in \mathcal{G}$ has entries e_{ij}, then $E(t_j) = \sum_1^r e_{ij} t_i$, and furthermore $E(t_1^{p_1} \dots t_r^{p_r}) = E(t_1)^{p_1} \dots E(t_r)^{p_r}$. This is called a *change of coordinates* in T. Let ψ_T be such a change of coordinates. Note that $\psi_T(Q) = Q$. We denote by ψ_Q its restriction on Q. We denote by ψ_S the change of variables (coordinates) in the polynomial ring S that is induced by ψ_Q. Therefore,

$$\varphi \psi_S = \psi_T \varphi \,.$$

If $f \in I = \mathrm{Ker}(\varphi)$, then $\psi_S(f) \in I$ since $\varphi\psi_S(f) = \psi_T\varphi(f) = 0$. Hence, $\psi_S(I) = I$.

Let G be an ideal in the polynomial ring T. If $H = \psi_T(G)$, then we have that

$$\begin{aligned} H_Q &= \psi_Q(G_Q), \\ H_R &= \psi_R(G_R), \\ H_S &= \psi_S(G_S), \end{aligned}$$

where the last equality follows from the fact that $\psi_S(I) = I$.

We say that H, H_Q, H_R, and H_S are *Type-A deformations* of G, G_Q, G_R, and G_S, respectively. Such a deformation preserves the Betti numbers.

3.4. Type-B deformations using initial ideals.

Let H be a graded ideal in the polynomial ring T, and let F be an initial ideal of H. By [Ba], we can choose a vector $\mathbf{z} = (z_1, \dots, z_n)$ with strictly positive integer coordinates, such that F is the initial ideal of H with respect to the weight order induced by the weight vector $\mathbf{z}$, cf. [Ei, Theorem 15.16]. Let $\tilde{H}$ be the homogenization of H in the polynomial ring $\tilde{T} = T[t]$; here $\tilde{T}$ is graded by $\deg(t_i) = z_i$ for $1 \le i \le r$ and $\deg(t) = 1$. Then t and

$t-1$ are regular elements on $\tilde{T}/\tilde{H}$, cf. [Ei, Theorem 15.17]. Set $\tilde{Q} = Q[t]$. Furthermore, set $\tilde{H}_{\tilde{Q}} = \tilde{H} \cap \tilde{Q}$. Clearly, $\tilde{H}_{\tilde{Q}} = \widetilde{H_Q}$.

We will show that t and $t-1$ are regular elements on $\tilde{Q}/\tilde{H}_{\tilde{Q}}$. Suppose that $t\tilde{h} \in \tilde{H}_{\tilde{Q}}$ for some $\tilde{h} \in \tilde{Q}$. It follows that $t\tilde{h} \in \tilde{H}$. Since t is a regular element on $\tilde{T}/\tilde{H}$, we conclude that $\tilde{h} \in \tilde{H}$. Hence, $\tilde{h} \in \tilde{H}_{\tilde{Q}} = \tilde{H}_{\tilde{Q}} \cap \tilde{H}$. Therefore, t is a non-zerodivisor on $\tilde{Q}/\tilde{H}_{\tilde{Q}}$. The argument for $t-1$ is similar. Suppose that $(t-1)\tilde{h} \in \tilde{H}_{\tilde{Q}}$ for some $\tilde{h} \in \tilde{Q}$. It follows that $(t-1)\tilde{h} \in \tilde{H}$. Since $t-1$ is a regular element on $\tilde{T}/\tilde{H}$, we conclude that $\tilde{h} \in \tilde{H}$. Hence, $\tilde{h} \in \tilde{H}_{\tilde{Q}} = \tilde{H}_{\tilde{Q}} \cap \tilde{H}$. Therefore, $t-1$ is a non-zerodivisor on $\tilde{Q}/\tilde{H}_{\tilde{Q}}$.

Furthermore, set $\tilde{R} = R[t]$. Let $\tilde{H}_{\tilde{R}} = \tilde{\varphi}_{\tilde{R}}^{-1}(\tilde{H}_{\tilde{Q}})$, where $\tilde{\varphi}_{\tilde{R}}$ is the isomorphism $\tilde{R} \cong \tilde{Q}$ induced by φ_R. We conclude that t and $t-1$ are regular elements on $\tilde{R}/\tilde{H}_{\tilde{R}}$.

Set $\tilde{S} = S[t]$, and let $\tilde{\varphi} : \tilde{S} \to \tilde{R}$ be the map induced by φ. Let $\tilde{H}_{\tilde{S}}$ be the preimage of $\tilde{H}_{\tilde{R}}$ in $\tilde{S}$. We will show that t and $t-1$ are regular elements on $\tilde{S}/\tilde{H}_{\tilde{S}}$. Suppose that $th \in \tilde{H}_{\tilde{S}}$ for some $h \in \tilde{H}_{\tilde{S}}$. It follows that $t\varphi(h) \in \tilde{H}_{\tilde{Q}}$. Since t is a non-zerodivisor on $\tilde{Q}/\tilde{H}_{\tilde{Q}}$, it follows that $\varphi(h) \in \tilde{H}_{\tilde{Q}}$. Hence, $h \in \varphi^{-1}(\tilde{H}_{\tilde{Q}}) = \tilde{H}_{\tilde{S}}$. Hence, t is a non-zerodivisor on $\tilde{S}/\tilde{H}_{\tilde{S}}$. The argument for $t-1$ is similar. Suppose that $(t-1)h \in \tilde{H}_{\tilde{S}}$ for some $h \in \tilde{H}_{\tilde{S}}$. It follows that $(t-1)\varphi(h) \in \tilde{H}_{\tilde{Q}}$. Since $t-1$ is a non-zerodivisor on $\tilde{Q}/\tilde{H}_{\tilde{Q}}$, it follows that $\varphi(h) \in \tilde{H}_{\tilde{Q}}$. Hence, $h \in \varphi^{-1}(\tilde{H}_{\tilde{Q}}) = \tilde{H}_{\tilde{S}}$. Hence, $t-1$ is a non-zerodivisor on $\tilde{S}/\tilde{H}_{\tilde{S}}$.

By construction we have that

$$T/F = \tilde{T}/\tilde{H} \otimes \tilde{T}/t \quad \text{and} \quad T/H = \tilde{T}/\tilde{H} \otimes \tilde{T}/(t-1).$$

It follows that

$$Q/F_Q = \tilde{Q}/\tilde{H}_{\tilde{Q}} \otimes \tilde{Q}/t \quad \text{and} \quad Q/H_Q = \tilde{Q}/\tilde{H}_{\tilde{Q}} \otimes \tilde{Q}/(t-1).$$

As $R \cong Q$,we get

$$R/F_R = \tilde{R}/\tilde{H}_{\tilde{R}} \otimes \tilde{R}/t \quad \text{and} \quad R/H_R = \tilde{R}/\tilde{H}_{\tilde{R}} \otimes \tilde{R}/(t-1).$$

Consider the grading of S with $\deg(x_i) = \deg(\varphi(x_i)) = \mathbf{a_i} \cdot \mathbf{z}$. Homogenize the ideal H_S using the variable t with respect to that grading. We denote this homogenization by $\widetilde{H_S}$. Recall that $\tilde{H}_{\tilde{S}}$ is the preimage of the ideal $\tilde{H}$ in $\tilde{S}$. Then we have $\widetilde{H_S} = \tilde{H}_{\tilde{S}}$. Therefore,

$$S/F_S = \tilde{S}/\tilde{H}_{\tilde{S}} \otimes \tilde{S}/t \quad \text{and} \quad S/H_S = \tilde{S}/\tilde{H}_{\tilde{S}} \otimes \tilde{S}/(t-1)\,.$$

Denote by $\tilde{\mathbf{F}}_{\tilde{R}}$ a graded minimal free resolution of $\tilde{R}/\tilde{H}_R$ over $\tilde{R}$. Note that this resolution is infinite. Then $\tilde{\mathbf{F}}_{\tilde{R}} \otimes \tilde{R}/t$ is a minimal free resolution of $R/F_R = \tilde{R}/\tilde{H}_{\tilde{R}} \otimes \tilde{R}/t$. Thus, the graded Betti numbers of R/F_R and $\tilde{R}/\tilde{H}_{\tilde{R}}$ coincide. On the other hand, $\tilde{\mathbf{F}}_R \otimes \tilde{R}/(t-1)$ is a non-minimal graded free resolution of $R/H_R = \tilde{R}/\tilde{H}_{\tilde{R}} \otimes \tilde{R}/(t-1)$. Therefore,

$$\tilde{\mathbf{F}}_R \otimes \tilde{R}/(t-1) \cong \mathbf{H}_R \oplus \mathbf{U}\,,$$

where $\mathbf{H}_R$ is a minimal graded free resolution of R/H_R and $\mathbf{U}$ is a trivial complex, cf. [Ei, Theorem 20.2]. The triviality of the complex $\mathbf{U}$ implies that the graded Betti numbers of R/F_R are obtained from those of $\tilde{R}/\tilde{H}_{\tilde{R}}$ by consecutive cancellations. Therefore, the graded Betti numbers of R/H_R are smaller or equal to those of R/F_R and are obtained by consecutive cancellations.

The same argument can be applied over S as follows. Denote by $\tilde{\mathbf{G}}_{\tilde{S}}$ a graded minimal free resolution of $\tilde{S}/\tilde{H}_{\tilde{S}}$ over $\tilde{S}$. Note that this resolution is finite. Then $\tilde{\mathbf{G}}_{\tilde{S}} \otimes \tilde{S}/t$ is a minimal free resolution of $S/F_S = \tilde{S}/\tilde{H}_{\tilde{S}} \otimes \tilde{S}/t$. Thus, the graded Betti numbers of S/F_S and $\tilde{S}/\tilde{F}_S$ coincide. On the other hand, $\tilde{\mathbf{G}}_{\tilde{S}} \otimes \tilde{S}/(t-1)$ is a non-minimal graded free resolution of $S/H_S = \tilde{S}/\tilde{H}_{\tilde{S}} \otimes \tilde{S}/(t-1)$. Therefore,

$$\tilde{\mathbf{G}}_S \otimes \tilde{S}/(t-1) \cong \mathbf{E}_S \oplus \mathbf{V}\,,$$

where $\mathbf{E}_S$ is a minimal graded free resolution of S/H_S and $\mathbf{V}$ is a trivial complex, cf. [Ei, Theorem 20.2]. The triviality of the complex $\mathbf{V}$ implies that the graded Betti numbers of S/F_S are obtained from those of $\tilde{S}/\tilde{H}_{\tilde{S}}$ by consecutive cancellations. Therefore, the graded Betti numbers of S/H_S are smaller or equal to those of S/F_S and are obtained by consecutive cancellations.

We say that F, F_Q, F_R and F_S are *Type-B deformations* of H, H_Q, H_R and H_S, respectively. A Type-B deformation increases the graded Betti numbers and the smaller Betti numbers can be obtained from the new greater ones by consecutive cancellations.

3.5. Type-C deformations using polarization.
Let X be a monomial ideal in T. Fix an $1 \leq i \leq r$. We recall the definition of the

partial polarization of X using the variable t_i. Let $\bar{T} = T[y]$. If m is a minimal monomial generator of X, then we set

$$\bar{m} = \begin{cases} \frac{m}{t_i} y & \text{if } t_i \text{ divides } m \\ m & \text{otherwise.} \end{cases}$$

Let $m_1, \ldots, m_l$ be the minimal monomial generators of X. Set

$$X_{pol} \;=\; (\bar{m}_1, \ldots, \bar{m}_l)\,.$$

Then

$$\bar{T}/\big(X_{pol} + (t_i - y)\big) = T/X.$$

The ideal X_{pol} in $\bar{T}$ is called the *partial polarization with respect to* t_i *of* X. Let $\alpha_1 t_1 + \ldots + \alpha_r t_r$ be a generic linear form in T (here $\alpha_1, \ldots, \alpha_r \in k$). Furthermore, let X' be the ideal in T such that

$$\bar{T}/\big(X_{pol} + (\alpha_1 t_1 + \ldots + \alpha_r t_r - y)\big) = T/X'\,.$$

Note that X' is usually not a monomial ideal. The elements $t_i - y$ and $\alpha_1 t_1 + \ldots + \alpha_r t_r - y$ are non-zerodivisors on $\bar{T}/X_{pol}$, since we use the usual construction of partial polarization in the polynomial ring T. Therefore, the ideals X and X' have the same Hilbert function. Let $\mathbf{dlex}_i$ be a degree-lex order in the polynomial ring T such that t_i is greatest. By construction, it follows that $\text{in}_{\mathbf{dlex}_i}(X') \supseteq X$. Since the two ideals have the same Hilbert function, we conclude that $\text{in}_{\mathbf{dlex}_i}(X') = X$. Therefore, the ideals X and X' are connected by a deformation over T, cf. [Ei, Chapter 15].

By 3.4, it follows that the ideals X, X_Q, X_R, and X_S are deformations of the ideals X', X'_Q, X'_R, and X'_S, respectively. We call these deformations *Type-C deformations.*

3.6. A proof of Theorem 1.6 (1) and (2).

We need the following observation from [GHP].

Lemma 3.6.1. *Let P and U be homogeneous ideals in R. Let $\tilde{P}$ and $\tilde{U}$ be the preimages of these ideals in S, respectively. The ideals P and U have the same Hilbert function over R, if and only if, the ideals $\tilde{P}$ and $\tilde{U}$ have the same Hilbert function over S.*

We are ready to prove the theorem.

Proof of Theorem 1.6: Let J be a graded ideal in R. Denote by ${}_SJ$ its preimage in S, and by ${}_QJ$ its isomorphic image in Q. Denote by ${}_TJ$ the ideal in T generated by ${}_QJ$. Also, set ${}_RJ = J$.

By Macaulay's Theorem 1.1 there exists a lex ideal L_T in the polynomial ring T with the same Hilbert function as ${}_TJ$. By [Ha, Pa], there exists a sequence of Type-A, Type-B, and Type-C deformations that connects ${}_TJ$ to L_T. The argument above implies that the ideal J is connected to L_R by a a sequence of Type-A, Type-B, and Type-C deformations, and also that the ideal J_S is connected to L_S by a a sequence of Type-A, Type-B, and Type-C deformations.

The ideal L_T is lex in the polynomial ring T. By Proposition 3.2.5 it follows that the ideal L_R is lex in the Veronese ring R. Thus, we have proved (1) and (2). □

3.7. A proof of Theorem 1.6 (3).

Let J be a graded ideal in R. We denote by $\beta^R_{i,j}(J)$ and $\beta^R_i(J)$ the graded Betti numbers and the total Betti numbers of J over R respectively.

Definition 3.7.1. A monomial ideal M in the polynomial ring T is called *Borel* if $mt_j \in M$ implies $mt_i \in M$ for all $1 \le i < j$. We say that a monomial ideal B in the Veronese ring R is *Borel* if ${}_TB$ is Borel.

Note that lex ideals are Borel. Borel ideals play an important role in Commutative Algebra since every generic initial ideal (assuming that $\mathrm{char}(k) = 0$) is Borel. The minimal free resolution of a Borel ideal over the polynomial ring S is the well known Eliahou-Kervaire resolution [EK]. It is proved using iterated mapping cones, cf. [PS3]. It yields a formula for the Betti numbers and shows that if a Borel ideal is generated in one degree then its resolution is linear. In this subsection, we prove analogues results for the Betti numbers over R in Theorem 3.7.4.

First, recall that a graded ideal J in R is said to have a *j-linear resolution* if $\beta^R_{i,i+s}(J) = 0$ for all i, s with $s \neq j$. In particular, J is generated in degree j in this case.

For a monomial m in the polynomial ring T, let $\max(m)$ (respectively, $\min(m)$) be the maximal (respectively, minimal) integer j such that t_j divides m. Set $F(1) = \{0\}$ and

$$F(i) = \{x \in S : x \text{ is a variable such that } \min(\varphi(x)) < i\} \quad \text{for } i = 2, 3, \dots, r.$$

Let $J_{F(i)}$ be the ideal in R generated by the image of $F(i)$ in R. The ideal $J_{F(i)}$ has a 1-linear resolution; indeed the following fact is known (cf. [HHR]).

Lemma 3.7.2. *Any ideal in R generated by variables has a 1-linear resolution.*

Let B be a Borel ideal in the Veronese ring R, and $\operatorname{mingens}({}_TB) = \{v_1, \ldots, v_p\}$ denote the set of its minimal monomial generators (note, that here each v_i is a monomial in T). We may assume that $\deg v_1 \leq \cdots \leq \deg v_p$ and that the ideals $(v_1, \ldots, v_j)$ are Borel for $j = 1, \ldots, p$. Fix monomial generators $u_1, \ldots, u_p$ of B such that $\varphi(u_j) = v_j$ for $j = 1, 2, \ldots, p$.

Lemma 3.7.3. *For $j = 1, 2, \ldots, p$, one has that*

$$((u_1, \ldots, u_{j-1}) :_R u_j) = J_{F(\max(v_j))}.$$

Proof: It is enough to prove the case $j = p$. Let B' be the ideal in R generated by $u_1, \ldots, u_{p-1}$. Then ${}_TB'$ is the Borel ideal in T generated by $v_1, \ldots, v_{p-1}$. Thus,

$$({}_TB' :_T v_p) = \big(\{t_i : i < \max(v_p)\}\big).$$

Since

$$({}_QB' :_Q v_p) = ({}_TB' :_T v_p) \cap Q = \big(\{\varphi(x) : x \in F\big(\max(v_p)\big)\}\big),$$

it follows that $(B' :_R u_p) = \varphi_R^{-1}({}_QB' :_Q v_p) = J_{F(\max(v_p))}$. $\square$

Theorem 3.7.4. *Let B be a Borel ideal in R. With the same notation as above, one has that*

$$\beta^R_{i,i+j}(B) = \sum_{v \in \operatorname{mingens}({}_TB),\ \deg v = qj} \beta^R_i\big(R/J_{F(\max(v))}\big) \quad \textit{for all } i, j.$$

If B is generated in degree j, then it's minimal free resolution over R is j-linear.

Proof: Note that $\deg v_j = q \deg u_j$ for all j. We use induction on p. Since $\max(v_1) = 1$ and since R is a domain, the statement is obvious if $p = 1$.

Suppose $p > 1$. Let $B' = (u_1, \ldots, u_{p-1})$. Consider the short exact sequence

$$0 \longrightarrow R/(B' :_R u_p)(-\deg u_p) \longrightarrow R/B' \longrightarrow R/B \longrightarrow 0,$$

where the first map is a multiplication by u_p. By Lemma 3.7.3 we have that $(B' :_R u_p) = J_{F(\max(v_p))}$. Let $\mathbf{G}$ be the graded minimal free resolution of $R/J_{F(\max(v_p))}(-\deg u_p)$ over R, and let $\mathbf{F}$ be that of R/B'. We will consider the mapping cone of the complex homomorphism $\mathbf{G} \to \mathbf{F}$ that is a lifting of the map $F/(B' :_R u_p)(-\deg u_p) \longrightarrow R/B'$. In the mapping cone construction, we have that the free module in homological degree i is $\mathbf{F}_i \oplus \mathbf{G}_{i-1}$. Since B' is Borel and $\deg u_p \geq \cdots \geq \deg u_1$, by induction hypothesis we have that the degree of any basis element of $\mathbf{F}_i$ is less than or equal to $\deg u_p + i - 1$. Also, since $J_{F(\max(v_p))}$ has a 1-linear resolution, the degree of each basis element of $\mathbf{G}_{i-1}$ is equal to $\deg u_p + i - 1$. These facts show that the mapping cone is a minimal free resolution of R/B. Then the theorem follows from the induction hypothesis. □

For a graded ideal J in R, we write $J_{\langle j\rangle}$ for the ideal in R generated by all elements of degree j in J. Let $\mathbf{m} = (x_1, \ldots, x_n) \subset R$ be the graded maximal ideal in R.

Corollary 3.7.5. *Let B be a Borel ideal in R. For all integers i, j, one has that*

$$\beta^R_{i,i+j}(B) = \beta^R_i(B_{\langle j\rangle}) + \beta^R_{i+1}(B_{\langle j-1\rangle}) - (\dim_k B_{j-1})\beta^R_{i+1}(k).$$

Proof: Note that $B_{\langle j\rangle}$ and $\mathbf{m}B_{\langle j-1\rangle}$ are Borel ideals generated in degree j. By Proposition 3.7.4 they have j-linear minimal free resolutions. Since

$$\big\{v \in \mathrm{mingens}({}_T B) : \deg(v) = qj\big\} = \mathrm{mingens}({}_T B_{\langle j\rangle}) \setminus \mathrm{mingens}\big({}_T(\mathbf{m}B_{\langle j-1\rangle})\big),$$

Proposition 3.7.4 implies that

$$\beta^R_{i,i+j}(B) = \beta^R_i(B_{\langle j\rangle}) - \beta^R_i(\mathbf{m}B_{\langle j-1\rangle}).$$

Consider the short exact sequence

$$0 \longrightarrow \mathbf{m}B_{\langle j-1\rangle} \longrightarrow B_{\langle j-1\rangle} \longrightarrow B_{\langle j-1\rangle}/\mathbf{m}B_{\langle j-1\rangle} \longrightarrow 0.$$

For each s, the short exact sequence leads to the long exact sequence

$$\begin{aligned} \ldots &\to \mathrm{Tor}^R_{i+1}(B_{\langle j-1\rangle}/\mathbf{m}B_{\langle j-1\rangle}, k)_s \\ &\to \mathrm{Tor}^R_i(\mathbf{m}B_{\langle j-1\rangle}, k)_s \to \mathrm{Tor}^R_i(B_{\langle j-1\rangle}, k)_s \to \mathrm{Tor}^R_i(B_{\langle j-1\rangle}/\mathbf{m}B_{\langle j-1\rangle}, k)_s \\ &\qquad \to \mathrm{Tor}^R_{i-1}(\mathbf{m}B_{\langle j-1\rangle}, k)_s \to \ldots \end{aligned}$$

Since the Veronese ring R is Koszul, the residue field k has a linear resolution over R. Thus both $B_{\langle j-1\rangle}/\mathbf{m}B_{\langle j-1\rangle} \cong \bigoplus_{k=1}^{\dim_k B_{j-1}} k$ and $B_{\langle j-1\rangle}$ have $(j-1)$-linear minimal free resolutions, and $\mathbf{m}B_{\langle j-1\rangle}$ has a j-linear minimal free resolution. Therefore, for each i, the long exact sequences on Tor yield the short exact sequence

$$0 \to \operatorname{Tor}_i^R(B_{\langle j-1\rangle}, k)_{i+j-1} \to \operatorname{Tor}_i^R(B_{\langle j-1\rangle}/\mathbf{m}B_{\langle j-1\rangle}, k)_{i+j-1} \to \operatorname{Tor}_{i-1}^R(\mathbf{m}B_{\langle j-1\rangle}, k)_{i-1+j} \to 0\,.$$

Hence we get

$$\beta_i^R(\mathbf{m}B_{\langle j-1\rangle}) = (\dim_k B_{j-1})\beta_{i+1}^R(k) - \beta_{i+1}^R(B_{\langle j-1\rangle})\,.$$

Then it follows that

$$\beta_{i,i+j}^R(B) = \beta_i^R(B_{\langle j\rangle}) - \beta_i^R(\mathbf{m}B_{\langle j-1\rangle}) = \beta_i^R(B_{\langle j\rangle}) + \beta_{i+1}^R(B_{\langle j-1\rangle}) - (\dim_k B_{j-1})\beta_{i+1}^R(k)$$

as desired. □

In the rest of this subsection we prove Theorem 1.6(3).

We have proved that Type-A deformations preserve the graded Betti numbers, and also that Type-B deformations increase the graded Betti numbers so that the smaller Betti numbers can be obtained from the new greater ones by consecutive cancellations. Since every generic initial ideal is Borel when $\operatorname{char}(k) = 0$, for any graded ideal J in R there exists a sequence of Type-A and Type-B deformations that connects ${}_TJ$ to some Borel ideal in T (cf. [Ei, Chapter 15.9]). Thus, to prove Theorem 1.6(3), it is enough to prove the following claim.

Proposition 3.7.6. *The graded Betti numbers of a Borel ideal B in R are smaller than or equal to those of the lex ideal L in R having the same Hilbert function as B.*

Proof: By Corollary 3.7.5, it is enough to show that $\beta_i^R(B_{\langle j\rangle}) \le \beta_i^R(L_{\langle j\rangle})$ for all i, j. Let $V = \{v_1, \ldots, v_s\}$ and $V' = \{v'_1, \ldots, v'_s\}$ be the sets of monomials of degree jq in ${}_TB$ and in ${}_TL$ respectively. By Green's Theorem [Gr] we have that

$$|\{v \in V : \max(v) \le \ell\}| \ge |\{v \in V' : \max(v) \le \ell\}|$$

for $\ell = 1, 2, \ldots, r$. Thus we may assume that $\max(v_\ell) \le \max(v'_\ell)$ for $\ell = 1, 2, \ldots, s$. Note that both ideals $J_{F(\max(v_\ell))}$ and $J_{F(\max(v'_\ell))}$ are Borel. Since $\operatorname{mingens}({}_TJ_{F(s)}) \subset \operatorname{mingens}({}_TJ_{F(s')})$, Theorem 3.7.4 implies that

$$\beta_i^R(R/J_{F(\max(v_\ell))}) \le \beta_i^R(R/J_{F(\max(v'_\ell))})$$

for all i. By applying Theorem 3.7.4 to $B_{\langle j\rangle}$ and $L_{\langle j\rangle}$, we obtain

$$\beta_i^R(B_{\langle j\rangle}) = \sum_{\ell=1}^{s} \beta_i^R(R/J_{F(\max(v_\ell))}) \leq \sum_{\ell=1}^{s} \beta_i^R(R/J_{F(\max(v'_\ell))}) = \beta_i^R(L_{\langle j\rangle})$$

for all i as desired. □

3.8. Green's Theorem.

We close this section by showing that Green's Theorem [Gr] holds over R.

Green's Hyperplane Restriction Theorem 3.8.1. *Let J be a graded ideal in R, and L be the lex ideal with the same Hilbert function as J. Let h be a generic linear form. For every $s \geq 0$ we have*

$$\dim_k \Big(R/(L,h)\Big)_s \geq \dim_k \Big(R/(J,h)\Big)_s .$$

Proof: Since h is a generic linear form in R, we have that $\varphi_R(h)$ is a generic form of degree q in the polynomial ring T. As in 3.7, let L_T be the lex ideal in T with the same Hilbert function as ${}_TJ$. A result of Gasharov [Ga] shows that

$$\dim_k \Big(T/(L_T, \varphi_R(h))\Big)_i \geq \dim_k \Big(T/({}_TJ, \varphi_R(h))\Big)_i$$

for every $i \geq 0$. Since $Q = T \cap Q$, $L_Q = L_T \cap Q$ and ${}_QJ = {}_TJ \cap Q$, and since $\varphi_R(h) \in Q$, we conclude that

$$\dim_k \Big(Q/(L_Q, \varphi_R(h))\Big)_{sq} \geq \dim_k \Big(Q/({}_QJ, \varphi_R(h))\Big)_{sq}$$

for every $s \geq 0$. Then the isomorphism $R \cong Q$ implies

$$\begin{aligned} \dim_k \Big(R/(L,h)\Big)_s &= \dim_k \Big(Q/(L_Q, \varphi_R(h))\Big)_{sq} \\ &\geq \dim_k \Big(Q/({}_QJ, \varphi_R(h))\Big)_{sq} = \dim_k \Big(R/(J,h)\Big)_s \end{aligned}$$

for every $s \geq 0$. □

4. Numerical versions of Macaulay's Theorem and Gotzmann's Persistence Theorem over Veronese rings

In this section we derive some corollaries of the proof of Theorem 1.6: a numerical characterization of the possible Hilbert functions in the Veronese ring R, and Gotzmann's Persistence Theorem over R.

Macaulay's Theorem 1.1 leads to a numerical criterion on what sequences of positive numbers are Hilbert functions. We will describe this criterion. Let i be a positive integer. It is easy to prove that for every $p \in \mathbf{N}$ there exist unique natural numbers $s_i > \ldots > s_1 \geq 0$ such that

$$p = \binom{s_i}{i} + \binom{s_{i-1}}{i-1} + \ldots + \binom{s_1}{1}. \tag{4.1}$$

This is called the *i'th Macaulay representation* of p. For example, the 3'rd Macaulay representation of 14 is $14 = \binom{5}{3} + \binom{3}{2} + \binom{1}{1}$. Set $0^{\langle i \rangle} = 0$ and

$$p^{\langle i \rangle} = \binom{s_i+1}{i+1} + \binom{s_{i-1}+1}{i} + \ldots + \binom{s_1+1}{2}.$$

For example, $14^{\langle 3 \rangle} = \binom{6}{4} + \binom{4}{3} + \binom{2}{2} = 20.$

Lex ideals are highly structured and it is easy to derive the inequalities characterizing their possible Hilbert functions. The following proposition is easy to prove.

Proposition 4.2. *Let $\alpha = \{\alpha_0 = 1, \alpha_1 = n, \alpha_2, \ldots\}$ be a sequence of non-negative integer numbers. If this sequence is the Hilbert function of a lex ideal L in the polynomial ring S, then $\alpha_{i+1} \geq \alpha_i^{\langle i \rangle}$ for each $i \geq 0$, and a strict inequality holds if and only L has a minimal monomial generator in degree $i+1$.*

By Macaulay's Theorem 1.1 it follows that:

Numerical version of Macaulay's Theorem 4.3. *Let* $\alpha = \{\alpha_0 = 1, \alpha_1 = n, \alpha_2, \ldots\}$ *be a sequence of non-negative integer numbers. This sequence is the Hilbert function of a graded ideal in the polynomial ring* S *if and only if* $\alpha_{i+1} \geq \alpha_i^{\langle i \rangle}$ *for each* $i \geq 0$.

Another important result on Hilbert functions in a polynomial ring is proved in [Go]:

Gotzmann's Persistence Theorem 4.4. *Let* J *be a graded ideal in the polynomial ring* S*. Set* $\alpha_i = \dim_k J_i$ *for* $i \geq 0$*. If* s *is such that* $\alpha_{s+1} = \alpha_s^{\langle s \rangle}$ *and* J *is generated in degrees* $\leq s$*, then* $\alpha_{i+1} = \alpha_i^{\langle i \rangle}$ *for each* $i \geq s$.

A similar Numerical version of Macaulay's Theorem holds over an exterior algebra. Gotzmann's Persistence Theorem also holds over an exterior algebra by [AHH]. We prove Theorem 4.5, which gives analogues of Theorems 4.3 and 4.4 over the q'th Veronese ring R.

We define a new operation $\langle q, i \rangle$ on the i'th Macaulay representation (4.1) as follows

$$p^{\langle q,i \rangle} = \binom{s_i + q}{i + q} + \binom{s_{i-1} + q}{i - 1 + q} + \ldots + \binom{s_1 + q}{1 + q}.$$

For example, $14^{\langle 2,3 \rangle} = \binom{7}{5} + \binom{5}{4} + \binom{3}{3} = 27$. Note that $p^{\langle i \rangle} = p^{\langle 1,i \rangle}$.

Theorem 4.5.

(1) **Numerical version of Macaulay's Theorem over Veronese rings.** *Let* $\alpha = \{\alpha_0 = 1,\ \alpha_1 = n,\ \alpha_2, \ldots\}$ *be a sequence of non-negative integer numbers. This sequence is the Hilbert function of a graded ideal in the* q*'th Veronese ring* R *if and only if* $\alpha_{i+1} \geq \alpha_i^{\langle q,i \rangle}$ *for each* $i \geq 0$.

(2) **Gotzmann's Persistence Theorem over Veronese rings.** *Let* J *be a graded ideal in the* q*'th Veronese ring* R*. Set* $\alpha_i = \dim_k J_i$ *for* $i \geq 0$*. If* s *is a number such that* $\alpha_{s+1} = \alpha_s^{\langle q,s \rangle}$ *and* J *is generated in degrees* $\leq s$*, then* $\alpha_{i+1} = \alpha_i^{\langle q,i \rangle}$ *for each* $i \geq s$.

Proof: Define a new sequence γ as follows:

$$\gamma_{j+1} = \begin{cases} \alpha_{\frac{j+1}{q}} & \text{if } j+1 \text{ is divisible by } q \\ \gamma_j^{\langle j \rangle} & \text{otherwise.} \end{cases}$$

Clearly, the sequence γ satisfies the equalities $\gamma_{j+1} = \gamma_j^{\langle j \rangle}$ for each $j \geq 0$ such that $j+1$ is not divisible by q. If $j+1$ is divisible by q, then the inequality $\gamma_{j+1} \geq \gamma_j^{\langle j \rangle}$ holds if and only if the inequality $\alpha_{i+1} \geq \alpha_i^{\langle q,i \rangle}$ holds for $i = \frac{j+1}{q} - 1$.

First, we will prove (1). Suppose that $\alpha_{i+1} \geq \alpha_i^{\langle q,i \rangle}$ for each $i \geq 0$. Therefore, $\gamma_{j+1} \geq \gamma_j^{\langle j \rangle}$ holds for all $j \geq 0$. By Theorem 4.3, γ is the Hilbert function of a lex ideal U in T. Set $L = U_R = \varphi^{-1}(Q \cap U)$. It is a lex ideal in R by Proposition 3.2.5, and its Hilbert function is given by the sequence α.

On the other hand, let $\alpha = \{\alpha_0 = 1,\ \alpha_1 = n,\ \alpha_2, \ldots\}$ be the Hilbert function of a graded ideal J in the Veronese ring R. By Theorem 1.6, α is the Hilbert function of a lex ideal L in R. Let U be the ideal generated by $\varphi(L)$ in the polynomial ring T. It is a lex ideal by Theorem 3.2.2. Since U is lex, it follows that the sequence γ is the Hilbert function of U. By Theorem 4.3, we have that $\gamma_{j+1} \geq \gamma_j^{\langle j \rangle}$ holds for all $j \geq 0$. Therefore, $\alpha_{i+1} \geq \alpha_i^{\langle q,i \rangle}$ for each $i \geq 0$. We have proved (1).

Now, we will prove (2). Denote by V the ideal in T generated by $\varphi(J)$. Let W be the lex ideal in T with the same Hilbert function as V. It follows that $U \subseteq W$, and $U_{iq} = W_{iq}$ for every $i \geq 0$. Denote by $\nu_i = \dim_k W_i$ for all i. Therefore, $\gamma_i \leq \nu_i$ for $i \geq 0$ and furthermore, $\gamma_{j+1} = \nu_{j+1}$ whenever q divides $j+1$.

Since we have $\alpha_{s+1} = \alpha_s^{\langle q,s \rangle}$ by assumption, it follows that $\gamma_{c+1} = \gamma_c^{\langle c \rangle}$ for $c = q(s+1) - 1$. Hence,

$$\nu_{c+1} = \gamma_{c+1} = \gamma_c^{\langle c \rangle} \leq \nu_c^{\langle c \rangle} \leq \nu_{c+1}\,,$$

where the last inequality holds by Macaulay's Theorem 4.3. Therefore,

$$\nu_{c+1} = \nu_c^{\langle c \rangle}\,.$$

Since J is generated in degrees $\leq s$, it follows that V is generated in degrees $\leq qs \leq q(s+1) - 1 = c$. By Gotzmann's Persistence Theorem 4.4 in the polynomial ring T, we have that $\nu_{j+1} = \nu_j^{\langle j \rangle}$ holds for all $j \geq c$. Hence, $\alpha_{i+1} = \alpha_i^{\langle q,i \rangle}$ for each $i \geq s$. □

5. Consecutive cancellations

Given a sequence of numbers $\{c_{i,j}\}$, we obtain a new sequence by a *cancellation* as follows: fix a j, and choose i and i' so that one of the numbers is odd and the other is even; then replace $c_{i,j}$ by $c_{i,j}-1$, and replace $c_{i',j}$ by $c_{i',j}-1$. We have a *consecutive cancellation* when $i'=i+1$. If we need to be specific, we call it a consecutive i,j-cancellation. The term "consecutive" is justified by the fact that we consider cancellations in Betti numbers of consecutive homological degrees.

A more precise version of Theorem 1.3 says:

Theorem 5.1. [Pe] *If J is a graded ideal in S and L is the lex ideal with the same Hilbert function, then the graded Betti numbers $\beta^S_{i,j}(S/J)$ can be obtained from the graded Betti numbers $\beta^S_{i,j}(S/L)$ by a sequence of consecutive cancellations.*

The analogue of this result holds over an exterior algebra.

Theorem 5.2. *Let E be an exterior algebra on variables $e_1,\dots,e_n$ over k. The ring E is graded by $\deg(e_i)=1$ for each i. If J is a graded ideal in E and L is the lex ideal with the same Hilbert function, then the graded Betti numbers $\beta^E_{i,j}(E/J)$ can be obtained from the graded Betti numbers $\beta^E_{i,j}(E/L)$ by a sequence of consecutive cancellations.*

The above theorem is not published, but it follows from Theorem 5.1 and a formula in [AAH] relating a finite monomial resolution over S and an infinite monomial resolution over E.

We have the following more precise versions of Theorem 1.6(3).

Theorem 5.3. *Let $R=S/I$ be a Veronese toric ring. If J is a graded ideal in R and L is the lex ideal with the same Hilbert function, then the graded Betti numbers $\beta^R_{i,j}(R/J)$ can be obtained from the graded Betti numbers $\beta^R_{i,j}(R/L)$ by a sequence of consecutive cancellations.*

Proof: By the proof of Theorem 1.6 it follows that we may assume that J is Borel. Set $d_{i,i+j}=\beta_{i+1}(L_{\langle j\rangle})-\beta_{i+1}(J_{\langle j\rangle})$. We showed in the proof of Proposition 3.7.6 that $d_{i,j}$ are nonnegative integers. Also, by Corollary 3.7.5 $\beta_{i,j}(J)=\beta_{i,j}(L)-d_{i-1,j}-d_{i,j}$. This formula and the non-negativity of $d_{i,j}$ show that the graded Betti numbers of J is obtained from those of L by a sequence of consecutive cancellations (a consecutive i,j-cancellation occurs $d_{i,j}$ times). □

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