

# Estimating Correlation Coefficient Between Two Complex Signals Without Phase Observation

Shigeki Miyabe<sup>1</sup>(✉), Notubaka Ono<sup>2</sup>, and Shoji Makino<sup>1</sup>

<sup>1</sup> University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8573, Japan  
{miyabe,maki}@tara.tsukuba.ac.jp

<sup>2</sup> National Institute of Informatics (NII)/The Graduate University for Advanced Studies (SOKENDAI), 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan  
onono@nii.ac.jp

**Abstract.** In this paper, we propose a method to estimate a correlation coefficient of two correlated complex signals on the condition that only the amplitudes are observed and the phases are missing. Our proposed method is based on a maximum likelihood estimation. We assume that the original complex random variables are generated from a zero-mean bivariate complex normal distribution. The likelihood of the correlation coefficient is formulated as a bivariate Rayleigh distribution by marginalization over the phases. Although the maximum likelihood estimator has no analytical form, an expectation-maximization (EM) algorithm can be formulated by treating the phases as hidden variables. We evaluate the accuracy of the estimation using artificial signal, and demonstrate the estimation of narrow-band correlation of a two-channel audio signal.

**Keywords:** Correlation · Complex signal · Maximum likelihood · EM algorithm

## 1 Introduction

Correlation of complex sequences plays an important role in array signal processing and almost all of its application [1]. The estimation of correlation is easily obtained by a simple product sum of the signal sequences. However, the correlation estimation by the product sum cannot be used when the phase observation is unreliable or unavailable. For example, in asynchronous distributed acoustic sensing systems which gather signals observed by multiple independent recording devices, the biases of the sampling frequencies of the individual devices cause drift of the phases [2, 3]. The effect of the drift on the amplitude is not serious, but the phase is strongly affected [4]. Also, when we analyze the output of the nonlinear signal processing in the amplitude domain, such as nonnegative matrix factorization (NMF) [5], the phases are often missing. Although many phase estimation methods are studied [6], the accurate estimation of the correlation cannot be guaranteed.

The goal of this paper is to estimate a correlation coefficient of two complex signal channels from the observation of amplitude without information of the

phase. Since the correlation coefficient is expressed by a nonnegative number, it cannot be used for estimation of phase difference, and the usage is somewhat limited. Still, the correlation coefficient gives important information, and the correlation estimation from amplitude is useful for specific purposes. For example, the estimation can be used to reduce the computational cost of the maximum likelihood compensation of drift [3], which requires large computational power and memory. As discussed in [3], only the limited frequency bins with high correlation contributes much to the observation, and estimation of the correlation only from the amplitude is informative for the efficient analysis discarding the useless frequency bins. Also, the correlation estimation from amplitude is expected to be useful for evaluation of channel capacity [7], for the estimation of SNR to optimize of the coefficients of the signal enhancement such as SS or Wiener filter, and so on.

To estimate a correlation coefficient between two complex random variables without phase observation, we propose a maximum likelihood estimation assuming that the original complex variables are generated from a zero-mean bivariate complex normal distribution. We show that the likelihood is given by the marginalization of the arguments, appearing as a *bivariate Rayleigh distribution* [7], whose parameter estimation algorithm has not been derived to the best of our knowledge. We derive an expectation-maximization (EM) algorithm to maximize the likelihood of the bivariate Rayleigh distribution, and obtain the maximum likelihood estimator of the correlation coefficient.

## 2 Statement of Problem

Suppose there are two correlated complex random variables  $X_i \in \mathbb{C}$ ,  $i = 1, 2$ , which have means of zero, variances  $\sigma_i^2$ , correlation  $\rho$  and the uniform arguments  $\Theta_i$  as

$$E[X_i] = 0, \quad (1)$$

$$E[|X_i|^2] = \sigma_i^2, \quad (2)$$

$$E[X_1 X_2^*] = \sigma_1 \sigma_2 \rho, \quad (3)$$

$$f_{\Theta_i}(\theta_i) = \frac{1}{2\pi}, \quad -\pi \leq \theta_i < \pi, \quad (4)$$

where  $E[\cdot]$  is the expectation operator,  $|\cdot|$  is the absolute value,  $\{\cdot\}^*$  is the complex conjugate,  $\sigma_i^2$  is the variance of  $X_i$ ,  $\Theta_i = \angle X_i$ ,  $\angle\{\cdot\}$  is the argument, and  $f_A(a)$  denotes the probability density of a random variable  $A$  whose sample is denoted as  $a$ . Hereafter, we denote random variables and the samples with upper and lower case letters, respectively.

Suppose the complex random variables  $X_1, X_2$  are unavailable, but we can observe their absolute values  $Y_1, Y_2$ :

$$Y_i = |X_i|. \quad (5)$$

Our goal is to estimate the correlation coefficient  $|\rho|$ , the absolute value of the cross correlation  $\rho$  between  $X_1$  and  $X_2$ , under the condition that only the absolute values are observed. It is obvious that the maximum likelihood estimator of the variance  $\sigma_i^2$  can be obtained as the average of the square of the observation:

$$\sigma_i^2 = E \left[ |X_i|^2 \right] \leftarrow \frac{1}{N} \sum_{n=1}^N y_i(n)^2, \quad (6)$$

where  $(n)$ ,  $n = 1, \dots, N$  denotes the index of the  $N$  observations. Note that we omit the sample index  $(n)$  when the explicit declaration is unnecessary. In contrast to the estimation of variance, the correlation coefficient  $|\rho|$  is different from the correlation of the absolute observations:

$$|\rho| = \frac{|E[X_1 X_2^*]|}{\sigma_1 \sigma_2} \neq \frac{E[Y_1 Y_2]}{\sigma_1 \sigma_2}. \quad (7)$$

Thus the following mean of the product of the observed absolute samples does not give a good estimation:

$$|\rho| \leftarrow \frac{1}{N \sigma_1 \sigma_2} \sum_{n=1}^N y_1(n) y_2(n). \quad (8)$$

Therefore, the estimation of the correlation coefficient with the absolute observation is not trivial.

### 3 Correlation Estimation Assuming Bivariate Complex Normal Distribution

#### 3.1 Probabilistic Model

In this section, we discuss the estimation of the correlation coefficient  $|\rho|$  from the absolute samples  $y_1, y_2$  assuming that the unobserved complex samples  $x$  with the statistics given by (1)–(3) are generated from a zero-mean bivariate complex normal distribution as

$$f_{X_1, X_2}(x_1, x_2; \rho) = \frac{\exp \left( -\frac{\sigma_2^2 |x_1|^2 + \sigma_1^2 |x_2|^2 - 2\sigma_1 \sigma_2 \operatorname{Re}[\rho^* x_1 x_2^*]}{\sigma_1^2 \sigma_2^2 (1 - |\rho|^2)} \right)}{\pi^2 \sigma_1^2 \sigma_2^2 (1 - |\rho|^2)}. \quad (9)$$

Then, the joint density of the observation  $Y_i$  and the unobserved argument  $\theta_1, \theta_2$  is expressed as

$$f_{Y_1, Y_2, \theta_1, \theta_2}(y_1, y_2, \theta_1, \theta_2; \rho) = \frac{y_1 y_2 \exp \left( -\frac{\sigma_2^2 y_1^2 + \sigma_1^2 y_2^2 - 2\sigma_1 \sigma_2 |\rho| y_1 y_2 \cos(\theta_1 - \theta_2 - \angle \rho)}{\sigma_1^2 \sigma_2^2 (1 - |\rho|^2)} \right)}{\pi^2 \sigma_1^2 \sigma_2^2 (1 - |\rho|^2)}, \quad (10)$$

by applying the polar coordinate conversion to (9). By the marginalization of the uniform distribution of the arguments  $\Theta_1, \Theta_2$ , the likelihood of the absolute observation  $Y_1, Y_2$  is given as a bivariate Rayleigh distribution, which can be found in many papers, e.g., [7] as a special case of multivariate Nakagami- $m$  distributions:

$$\begin{aligned}
 f_{Y_1, Y_2}(y_1, y_2; |\rho|) &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f_{Y_1, Y_2, \Theta_1, \Theta_2}(y_1, y_2, \theta_1, \theta_2; \rho) d\theta_1 d\theta_2 \\
 &= \frac{4y_1 y_2}{\sigma_1^2 \sigma_2^2 (1 - |\rho|^2)} I_0 \left( \frac{2|\rho| y_1 y_2}{\sigma_1 \sigma_2 (1 - |\rho|^2)} \right) \exp \left( -\frac{\sigma_2^2 y_1^2 + \sigma_1^2 y_2^2}{\sigma_1^2 \sigma_2^2 (1 - |\rho|^2)} \right),
 \end{aligned} \tag{11}$$

where  $I_\nu(\cdot)$  denotes the modified Bessel function of the first kind with the order  $\nu$ . It can be seen that the density of  $Y_1$  and  $Y_2$  depends on the correlation coefficient  $|\rho|$  but not on the argument  $\angle \rho$  of the correlation. Therefore, the maximization of the likelihood gives the estimation of the correlation coefficient  $|\rho|$ . However, the maximum likelihood estimator does not have the analytical form.

### 3.2 Maximum Likelihood Estimation by EM Algorithm

Here we describe the maximum likelihood estimation of the correlation coefficient  $|\rho|$  by the iterative procedure. By treating the observed samples  $y_1(n)$  and  $y_2(n)$  together with the unobserved arguments  $\theta_1(n)$  and  $\theta_2(n)$ , we can formulate the EM algorithm. We treat the arguments  $\Theta_1, \Theta_2$  as hidden variables, and the posterior density of the hidden variables is given by

$$\begin{aligned}
 f_{\Theta_1, \Theta_2 | Y_1, Y_2}(\theta_1, \theta_2 | y_1, y_2; \rho) &= \frac{f_{Y_1, Y_2, \Theta_1, \Theta_2}(y_1, y_2, \theta_1, \theta_2; \rho)}{f_{Y_1, Y_2}(y_1, y_2; \rho)} \\
 &= \frac{\exp \left( \frac{2|\rho| y_1 y_2 \cos(\theta_1 - \theta_2 - \angle \rho)}{\sigma_1 \sigma_2 (1 - |\rho|^2)} \right)}{2\pi I_0 \left( \frac{2|\rho| y_1 y_2}{\sigma_1 \sigma_2 (1 - |\rho|^2)} \right)}.
 \end{aligned} \tag{12}$$

Then, the auxiliary function  $Q(|\rho|, |\bar{\rho}|)$  to maximize in each iteration of the EM algorithm is obtained as

$$\begin{aligned}
 Q(|\rho|; |\bar{\rho}|) &= \sum_{n=1}^N \langle \log f_{Y_1, Y_2, \Theta_1, \Theta_2}(y_1(n), y_2(n), \theta_1(n), \theta_2(n); \rho) \rangle_{\theta_1(n), \theta_2(n) | y_1(n), y_2(n); |\bar{\rho}|} \\
 &\propto \frac{2|\rho|}{\sigma_1 \sigma_2 (1 - |\rho|^2)} \sum_{n=1}^N y_1(n) y_2(n) \lambda(n) - N \log (1 - |\rho|^2),
 \end{aligned} \tag{13}$$

$$\langle g(a) \rangle_{a|b;c} = \int_{\mathcal{D}(a)} f_{A|B}(a|b; c) g(a) da, \tag{14}$$

where  $\mathcal{D}(a)$  is the domain of the variable  $a$ , and  $\lambda(n)$  is the posterior expectation of  $\cos(\theta_1(n) - \theta_2(n) - \angle\rho)$  with the current parameter estimation  $|\bar{\rho}|$ , given by

$$\begin{aligned} \lambda(n) &= \langle \cos(\theta_1(n) - \theta_2(n) - \angle\rho) \rangle_{\theta_1(n), \theta_2(n) | y_1(n), y_2(n); |\bar{\rho}|} \\ &= L \left( \frac{2|\bar{\rho}| y_1(n) y_2(n)}{\sigma_1 \sigma_2 (1 - |\bar{\rho}|^2)} \right), \end{aligned} \quad (15)$$

$$L(x) = I_1(x) / I_0(x) \quad (16)$$

Since  $L(x)$  is a monotonically increasing function giving  $L(0) = 0$  and  $\lim_{x \rightarrow \infty} L(x) = 1$ ,  $\lambda(n)$  acts as weighting in the update of the estimation of  $|\rho|$  in the M-step:

$$|\rho| = \frac{1}{N\sigma_1\sigma_2} \sum_{n=1}^N y_1(n) y_2(n) \lambda(n). \quad (17)$$

By iterating the updates of E- and M-steps given by (15) and (17), respectively, the estimation of  $|\rho|$  converges to a local optimal. Note that  $\sigma_1^2$  and  $\sigma_2^2$  are estimated by (6), which can also be derived as the maximization of the auxiliary function, although the related terms are omitted in (13). Also note that  $L(x)$  can be calculated by one-dimensional table lookup.

## 4 Experimental Results

### 4.1 Evaluation with Artificial Signal

To evaluate the performance of the proposed method, we conducted a numerical simulation to estimate the correlation coefficients of artificial two-channel complex signals generated from pseudorandom numbers. To show the baseline, we also evaluated the performance of absolute correlation given by (8). In addition, to show the upper limit with the ideal condition, we evaluated the standard maximum likelihood estimation under the condition where the original complex sequences is available, given by

$$|\rho| \leftarrow \frac{1}{N\sigma_1\sigma_2} \left| \sum_{n=1}^N x_1(n)^* x_2(n) \right|. \quad (18)$$

As the evaluation criterion, we calculated the root mean squared errors (RMSEs).

The correlated data are generated by the linear mixture of the two independent pseudorandom numbers with the same variance. We controlled the correlation by manipulating the linear mixture. To evaluate the robustness of the proposed method against the mismatch of the Gaussian assumption, we also evaluated the artificial signals generated by super-Gaussian pseudorandom numbers. The super-Gaussian data are generated from the circular generalized normal distribution [8], whose density of the random variables  $X_1, X_2$  is given by

$$f_{X_i}(x_i) = \frac{c_i \exp\left(-\left(\frac{|x_i|}{\xi_i}\right)^{c_i}\right)}{2\pi \xi_i^2 \Gamma\left(\frac{2}{c_i}\right)}, \quad (19)$$

where  $\xi_i > 0$  is the scale parameter,  $c_i > 0$  is the shape parameter, and  $\Gamma(\cdot)$  is a gamma function. The shape parameters are set as  $c_1 = 0.5$  and  $c_2 = 0.8$ , and scale parameters are adjusted to give the unit variance. With such shape parameter setting, the sequences have super Gaussian property with long tail, and their kurtosis, a well-used Gaussianity measure, are about 7.4 and 2.3. Note that the kurtosis is changed after the mixing to give correlation.

We show an example of the estimation in Fig. 1. For both Gaussian and super-Gaussian data, the absolute correlation does not give accurate estimation. Although the variance of the proposed method is slightly larger than the estimation with arguments, the estimated correlation coefficients distribute near the true correlation coefficients and the accuracy of the proposed method is match better than the baseline. The estimation of the correlation coefficients of super-Gaussian data tends to underestimate the correlation, but the RMSEs are similar to the Gaussian case without the model mismatch. Thus it is confirmed that the proposed method can effectively estimate the correlation coefficients of the complex sequences without the observation of arguments.

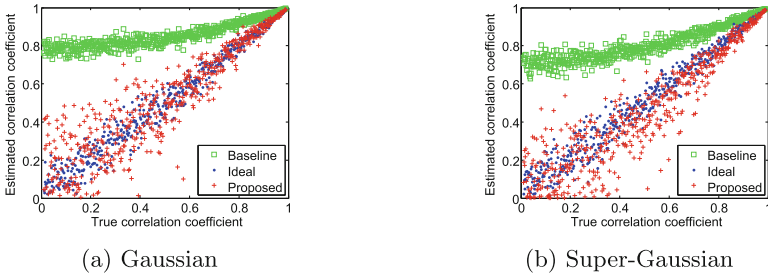
To examine the effect of the bias and variances of the estimation, we evaluate RMSEs of various numbers of observed samples. The result is shown in Fig. 2. Although the proposed method has larger variance than the ideal estimation with phase, we can see that the estimation accuracy of the proposed method improves according to the increase of the number of the samples when the data is Gaussian. Thus the proposed method can estimate the correlation of Gaussian data effectively with small bias. However, we can see the saturation of the improvement of the accuracy under the condition mismatch with the super-Gaussian data.

## 4.2 Demonstration with Audio Data

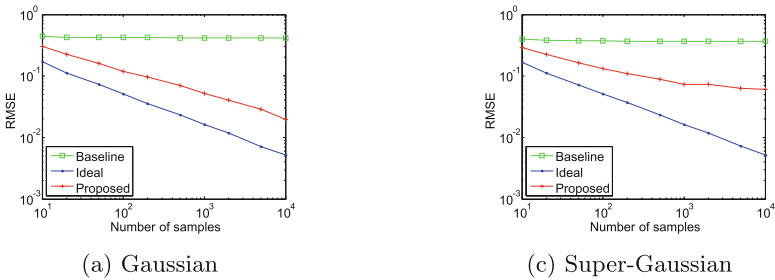
As a demonstration of the correlation coefficient estimation in practical signal processing, we evaluated the estimation of the correlation coefficients of the narrow band amplitudes of a two-channel audio signal.

We analyzed observation of speech mixture by an array of two microphones. The data is chosen from the Underdetermined Test dataset of the Signal Separation Evaluation Campaign (SiSEC) [9], a benchmark of speech separation. Utterances of four female speakers were recorded in a room whose reverberation time  $T_{60}$  is 250 ms. The spacing of the microphones was 1 m. The recorded data was 10 s long, sampled with the frequency of 16 kHz. The signal was analyzed by short-time Fourier transform with the von Hann window of the length 1024 samples and the shift 128 samples. The number of the frames is 1258.

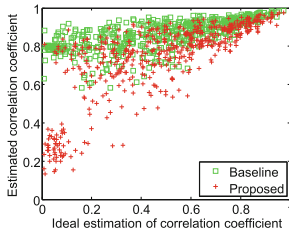
The estimated results are shown in Fig. 3. In contrast to the artificial signals, the true correlation coefficients are unknown. Thus it should be noted that the horizontal axis is not the true correlation coefficients but the estimated correlation coefficients by the ideal estimation in (18) with the original complex signal given. We can see that the proposed method is much better than the baseline. Superiority of the proposed method holds for other frame lengths, which strongly



**Fig. 1.** An example of the estimation results of correlation coefficients between two artificial signals generated from (a) Gaussian and (b) super-Gaussian pseudorandom numbers. The number of samples was 100 for each trial. The EM iteration number was 10. The RMSEs are about 0.42, 0.05 and 0.12 for the baseline, the ideal estimation and the proposed method, respectively for the results in (a), and about 0.05, 0.37 and 0.12 for the results in (b).



**Fig. 2.** Root mean squared errors for the number of the samples  $N = 10, 20, 50, 100, 200, 500, 1000, 2000, 5000, 10000$ . The RMSEs of each condition was calculated from 100 trials. The EM iteration number was set to 200.



**Fig. 3.** Estimation results of narrow band correlation coefficients of a two-channel audio signal. The EM iteration number was 10. The RMSEs from the ideal estimation are about 0.28 and 0.20 for the baseline and the proposed method, respectively.

affects the correlation. However, the accuracy is not as good as that of the artificial signal. Thus, further analysis is required to improve the estimation accuracy with the realistic signals.

## 5 Conclusions

In this paper, we proposed an EM algorithm to obtain the maximum likelihood estimation of the correlation coefficient of the two correlated complex random variables only with the observation of the absolute values. Assuming that the original complex data are generated from a zero-mean bivariate complex normal distribution, we formulated the likelihood of the correlation coefficient. Although the maximum likelihood estimator is not analytical, we formulated the EM algorithm by treating the difference of the arguments as a hidden variable. We evaluated accuracy and the robustness against the model mismatch of the proposed method by the simulation using artificial signals. Also, we demonstrated the estimation of the narrow-band correlation coefficients of the two-channel audio signal. It is confirmed that the proposed method is much more accurate than the estimation with the correlation of the absolute values. However, estimation of the real signal was worse than that of the artificial signal, and the further analysis and improvement are required. Our future work includes the modified maximum likelihood estimation with a non-Gaussian distribution.

**Acknowledgment.** This work was supported by JSPS KAKENHI Grant Number 23240023.

## References

1. Johnson, D., Dudgeon, D.: *Array Signal Processing: Concepts and Techniques*. Simon & Schuster, New York (1992)
2. Markovich-Golan, S., Gannot, S., Cohen, I.: Blind sampling rate offset estimation and compensation in wireless acoustic sensor networks with application to beamforming. In: *Proceedings of IWAENC*, pp. 1–4 (2012)
3. Miyabe, S., Ono, N., Makino, S.: Blind compensation of interchannel sampling frequency mismatch for ad hoc microphone array based on maximum likelihood estimation. *Signal Process.* **107**(2015), 185–196 (2015)
4. Chiba, H., Ono, N., Miyabe, S., Takahashi, Y., Yamada, T., Makino, S.: Amplitude-based speech enhancement with nonnegative matrix factorization for asynchronous distributed recording. In: *Proceedings of IWAENC*, pp. 204–208 (2014)
5. Lee, D., Seung, H.: Algorithms for non-negative matrix factorization. In: *Proceedings of NIPS*, pp. 556–562 (2001)
6. Griffin, D., Lim, J.: Signal estimation from modified short-time Fourier transform. *IEEE Trans. ASSP* **32**(2), 236–243 (1984)
7. Fraidenraich, G., L ev eque, O., Cioffi, J.: On the MIMO channel capacity for the nakagami- $m$  channel. *IEEE Trans. Inf. Theory* **54**(8), 3752–3757 (2008)
8. Novey, M., Adali, T., Roy, A.: A complex generalized Gaussian distribution-characterization, generation and estimation. *IEEE Trans. Signal Process.* **58**(3), 1427–1433 (2010)
9. Ono, N., Koldovsky, Z., Miyabe, S., Ito, N.: The 2013 Signal Separation Evaluation Campaign. In: *Proceedings of MLSP*, pp. 1–6 (2013)