Non-Euclidean Geometry: A mathematical revolution during the long 19th century

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History of Mathematics
Introduction

Saccheri

Lobachevskii

Poincaré
What is geometry?

Until the 19\textsuperscript{th} century, geometry was the study of the mathematical properties of \textit{figures}. All figures were regarded as embedded in Euclidean space. (Ex. Spherical geometry.)

During the 19\textsuperscript{th} century, geometry became the study of physical and mathematical \textit{space itself}. It was an investigation of the properties of the types of spaces \textit{in which} figures could be found. \textbf{Hyperbolic space} was put forward as a viable alternative to \textbf{Euclidean space}. Geometers realized that \textbf{elliptical} (ex. \textit{spherical}) spaces, and many more, were also possible.

By the end of the century, geometry had come to be seen as the abstract investigation of the consequences of some set of axioms. Space was understood as the set of points determined by these axioms.
Was there a revolution in the study of geometry?

Since antiquity, mathematicians had sought to demonstrate the parallel postulate, but by the beginning of the 19\textsuperscript{th} century, there was a growing anxiety that it could not be done. (Crisis.)

At the beginning of the 19\textsuperscript{th} century, three men, independently, arrived at a geometry of hyperbolic space (Gauss, Bolyai, Lobachevsky). In the middle of the century, Riemann showed that there are an infinite number of non-Euclidean geometries, and pointed out that elliptical spaces are possible.

By the end of the century, mathematicians were no longer asking, “what is the correct geometry?” but, rather, trying to determine what sets of axioms would produce what kinds of spaces. . .
Topics in the non-euclidean revolution

We will look at:

- An extended attempt to prove that euclidean space is the only “correct” space. (We can now see this as logically flawed.)
- An abstract definition of parallel lines, which leads to a new kind of space – hyperbolic space.
- A model for showing that a hyperbolic plane is logically consistent with, or mathematically mappable to, a euclidean plane.
Giovanni Girolamo Saccheri (1667–1733)

Saccheri was an Italian Jesuit, who worked as professor of philosophy, theology and mathematics at Turin and Pavia.

In 1733, he published *Euclides ab omni naevo vindicatus* (Euclid freed from all flaws), the first part of which was an attempt to demonstrate Euclid’s 5\(^{th}\) postulate. (The other “flaws” concern fourth proportionals and compound ratios.)

- Many of Saccheri ideas were predated, and perhaps influenced, by the work of Omar Khayyam and Ibn al-Haytham.
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Saccheri

The Saccheri quadrilateral

Hypothesis of the Right Angle (HRA) : $\angle C = \angle D = 90^\circ$

Hypothesis of the Obtuse Angle (HOA) : $\angle C = \angle D > 90^\circ$

Hypothesis of the Acute Angle (HAA) : $\angle C = \angle D < 90^\circ$
The structure of the “proof”

Prop. 3:

\[ \text{HRA} \implies AB = CD \implies \text{\(\angle\)s of } \triangle = 180^\circ \]
\[ \text{HOA} \implies AB > CD \implies \text{\(\angle\)s of } \triangle > 180^\circ \]
\[ \text{HAA} \implies AB < CD \implies \text{\(\angle\)s of } \triangle < 180^\circ \]

The entire first part of *Euclides vindicatus* is structured like an extended indirect proof:

- A): He uses HRA to demonstrate the parallel postulate.
- B): He uses HOA to show that the geometry so created is inconsistent with *Elem.* I 16 & 17.\(^1\)
- C): He uses HAA to argue that it leads to consequences that are “repugnant to the nature of the straight line.”

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\(^1\) *Elem.* I 16: an exterior \(\angle\) of a \(\triangle\) is greater than the sum of the 2 interior \(\angle\)s. *Elem.* I 17: the sum of 2\(\angle\)s of a \(\triangle\) are less than 2\(R\).
The hypothesis of the obtuse angle

- **EV Props. 11 & 12 (HRA, HOA):** If a line falls on two given lines such that it is perpendicular to one and makes an acute angle with the other, then the two given lines will meet in the direction of the said acute angle. (If $\angle LPA$ is right and $\angle PAD$ is acute, then $AD$ will intersect $PL$.)

- **EV Props. 13 (HRA, HOA):** If a line falls on two given lines such that it makes the two interior angles on the same side less than two right angles, then the two lines will meet in the direction of the said interior angles. (If $\angle AXL + \angle XAD < 2R$, then $AD$ will meet $XL$.)
  - Contradiction with *Elem.* I.17.

- **EV Props. 14 (HOA):** “The hypothesis of the oblique angle is absolutely false, because it destroys itself.”
The hypothesis of the acute angle

▶ **EV Prop. 32 (HAA):** If HAA is true, there will exist a line with the following properties:
  ▶ $AX$ is a limit to the set of lines passing through point $A$ that meet $BX$ and also a limit to the set of lines passing through point $A$ that have two distinct perpendiculars with line $BX$ – one in each direction.
  ▶ $AX$ meets $BX$ at one point, infinitely distant.
  ▶ $AX$ is asymptotic to $BX$.\(^2\)
  ▶ $AX$ is a *straight line*.

▶ **EV Prop. 33 (HAA):** “The hypothesis of the acute angle is absolutely false, *because it is repugnant to the nature of a straight line.*”

\(^2\)That is the distance between the two lines will become less than any given value.
Nikolai Ivanovich Lobachevsky (1792–1856)

- Born to a provincial, middle class family. His father died when he was 8.
- Graduated from Kazan University with a degree in physics and mathematics.
- Worked at Kazan University his whole life.
- He developed his ideas on non-Euclidean geometry from 1826 to 1855.
- Died in poverty.
Lobachevskii begins by defining a line, \( \ell_p \), through a given point \( P \), as parallel to a given line, \( \ell_0 \), which is joined to \( P \) by \( p \), when \( \ell_p \) divides all the lines that pass through point \( P \) into two mutually exclusive sets, those which are intersecting, \( \ell_i \), and those which are non-intersecting, \( \ell_n \). (\( \ell_p \parallel \ell_0 \).) That is, where the angle at \( P \) is written \( \Pi(p) \):

- lines, \( \ell_i \), where \( \alpha_1 < \Pi(p) \), intersect \( \ell_0 \), and
- lines, \( \ell_n \), where \( \alpha_2 > \Pi(p) \), do not intersect \( \ell_0 \).

Where \( \Pi(p) = \pi/2 = 90^\circ \), then there is a unique parallel, which is parallel in both directions.

Where \( \Pi(p) < \pi/2 = 90^\circ \), then there is another line on the opposite side at the same angle that is parallel in the opposite direction.

With respect to \( \ell_0 \), all lines through \( P \) can be classified as 1) intersecting, 2) parallel, or 3) non-intersecting.
Properties of the hyperbolic plane, $H^2$

- The sum of the angles of a triangle are less than $180^\circ = \pi$.
- There is no distinction between similarity and congruency.
  (There are four congruency theorems SAS, SSS, AAS, and AAA.)
  - As triangles get larger and smaller, the angles also change.
- There are no straight lines everywhere equidistant from one another.
- Lines parallel to the same line need not be parallel to one another. (Parallelism is non-transitive, because of directionality.)
- Two lines which intersect one another may both be parallel to the same line. (Again, because of directionality.)
- and so on. . .
A model of the Euclidean plane in hyperbolic space

Lobachevskii then constructed a three dimensional model of Euclidean space within his hyperbolic space. (Just as we can construct a three dimensional model of spherical space within Euclidian space.)

- He shows that the planes containing three non-coplanar parallel lines contain solid angles that sum to 180°. (Prism theorem.)
- He develops a horocycle as a curve perpendicular to a set of coplanar parallel lines, and a horosphere as the solid of rotation of a horocycle.
- He shows that the geometry of the horosphere is Euclidean by mapping it to the Euclidean plane. In this way, he shows that a Euclidean plane can be embedded in hyperbolic space – just as a spherical plane can be embedded in Euclidean space.
Jules Henri Poincaré (1854–1912)

- Born to an influential, bourgeois family.\(^3\)
- Graduated from the École Polytechnique and the École des Mines.
- He eventually became a professor at the Sorbonne.
- He was an influential figure in French mathematical sciences and also wrote many popular works.

\(^3\)His sister married the philosopher Emile Broutroux and his cousin Raymond Poincaré became president of France.
Some axioms (Hilbert, *Grundlagen der Geometrie*, 1899)

- **Incidence.1:** For every two points $A, B$ there exists a line $a$ that contains each of the points $A, B$.
- **Incidence.2:** For every two points $A, B$ there exists no more than one line that contains each of the points $A, B$.
- **Congruence.2:** If a segment $A'B'$ and a segment $A''B''$ are congruent to the same segment $AB$, then segments $A'B'$ and $A''B''$ are congruent to each other (*transitivity*).
- **Congruence.5:** If for two triangles $ABC$ and $A'B'C'$ the congruences $AB \cong A'B', AC \cong A'C'$ and $\angle BAC \cong \angle B'A'C'$ are valid, then the congruence $\triangle ABC \cong \triangle A'B'C'$ is also satisfied. (Used to show SAS congruence, *Elements* I.4.)
- and so on. . .
Preliminary, polar inverse of a point by a circle

The polar inverse of a point\(^4\) with respect to a circle, or a pair of orthogonal circles is found as follows. Where \(A\) is an internal point, join \(OA\) and erect \(\perp AP\). Join \(OP\) and draw tangent \(PB\) to meet \(OA\) extended to \(B\). Any circle through \(A\) and \(B\) is \(C' \perp C\).

The same construction can be used when an external point, \(B\), is given. Find the tangent \(BP\) and drop \(\perp PA\).

\(^4\)Polar inversion is defined by the relation \(AO \cdot BO = r^2\).
The Poincaré disk

The hyperbolic plane, $\mathcal{H}^2$, is defined as the set of points of the disk inside circle $\Gamma$, but not including the circumference itself.

A $P$-point is any point inside the circle.

A $P$-line is any circle orthogonal to $\Gamma$, or a line drawn through the center of the circle. (A line or circle $\perp \Gamma$)

The Euclidean points, lines and circle are simply referred to as such.
Consistency with the axioms of Euclidean geometry

- We can use the model to demonstrate all of Hilbert’s axioms of Euclidian geometry. For example, we show that two P-points always determine a unique P-line (Incidence 1 & 2).

- Suppose we have two P-points, A and B.
  - a) If line $AB$ goes through center $O$, then $ABO$ is the only line through $A$ and $B$, and it is also a P-line.
  - b) If not, we find the polar inverse of $A$ as $A'$ and draw circle $\gamma$ through $A'$, $A$ and $B$. Then $\gamma$ is the only circle through $A'$, $A$ and $B$ and it is orthogonal to $\Gamma$.

- Therefore, there is always a unique P-line through any two P-points.

- We can do similar proofs for all of the Euclidean axioms, except the parallel axiom.
Parallels in the Poincaré disk

Hilbert’s parallel axiom: “Let a be any line and A a point not on it. Then there is at most one line in the plane that contains a and A that passes through A and does not intersect a.”

- Two P-lines that are $P$-parallel intersect circle $\Gamma$ at the same point. (Hence, they are tangent circles.)

- There are many P-lines through a given P-point that do not intersect a given P-line. Proof by construction. (We let $\gamma$ be a given P-line and $A$ a given P-point. We find the inverse of $A$, as $A'$. . . )

- The two angles of parallelism, $\Pi(p)$, associated with a given P-point and P-line are equal. Proof by contradiction. (Let $AMB$ be a P-line with two parallels, $AD$ and $DB$ through P-point $D$. Assume, to the contrary that, $\angle ADM > \angle BDM$. . . )
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A Saccheri quadrilateral in the Poincaré disk
Overview

In the 18th century, it became increasingly clear that it would not be possible to prove that Euclidean geometry was the only valid geometry. We looked at Saccheri’s failed attempt to prove this.

In the 19th century, there were a number of attempts to develop non-Euclidean geometries and to show that these were valid. Mathematicians became increasingly concerned with validity as opposed to truth, and with modeling one type of geometry in another.

Around the turn of the 20th century, there was new foundational work on Euclidean geometry. This lead, in turn, to new work on showing the logical consistency between various types of geometries. The mechanism for this was a map, or model.