

A large complete graph in a space contains a link with large link invariant

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Abstract Let k be a non-negative integer. Then any embedding of the complete graph on $6 \cdot 2^k$ vertices into a three-space contains a two-component link with the absolute value of its linking number greater than or equal to 2^k . Let j be a non-negative integer. Then any embedding of the complete graph on $48 \cdot 2^j$ vertices into a three-space contains a knot with the absolute value of the second coefficient of its Conway polynomial greater than or equal to 2^{2j} .

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§1. Introduction

Throughout this paper we work in the piecewise linear category. Let m be a natural number. By K_m we denote the complete graph on m vertices. Let G be a finite graph. A *cycle* of G is a subgraph of G that is homeomorphic to a circle. A cycle on n vertices is called an *n -cycle*. Let $L = J_1 \cup J_2$ be an oriented two-component link in the three-dimensional Euclidean space R^3 . By $lk(L) = lk(J_1, J_2)$ we denote the linking number of L . Even when $L = J_1 \cup J_2$ is not oriented the absolute linking number $|lk(L)| = |lk(J_1, J_2)|$ is well-defined. Let J be a knot in R^3 . By $a_2(J)$ we denote the second coefficient of the Conway polynomial of J .

In [1] Conway and Gordon showed the following theorem.

Theorem 1. [1] (1) *Let $f : K_6 \rightarrow R^3$ be an embedding. Then there are mutually disjoint 3-cycles γ_1 and γ_2 of K_6 such that $|lk(f(\gamma_1), f(\gamma_2))| \geq 1$.*

(2) *Let $f : K_7 \rightarrow R^3$ be an embedding. Then there is a 7-cycle γ of K_7 such that $|a_2(f(\gamma))| \geq 1$.*

In [2] Flapan extended this result as follows.

Theorem 2. [2] (1) *Let n be a natural number. Let $m = n(15n - 9)$. Let $f : K_m \rightarrow R^3$ be an embedding. Then there are mutually disjoint cycles γ_1 and γ_2 of K_m such that $|lk(f(\gamma_1), f(\gamma_2))| \geq n$.*

(2) *Let n be a natural number. Let p be a natural number with $p \geq 4\sqrt{n}$, and let $m = 2p(15p - 9)$. Let $f : K_m \rightarrow R^3$ be an embedding. Then there is a cycle γ of K_m such that $|a_2(f(\gamma))| \geq n$.*

By Theorem 2 (1) we have that for any natural number n there is a natural number m such that every embedding of K_m into R^3 contains a link whose absolute linking number is greater than or equal to n . In [2] m is given by $m = n(15n - 9)$. In this paper we show that $m = 12n$ is sufficient. By Theorem 2 (2) we have that for any natural number n there is a natural number m such that every embedding of K_m into R^3 contains a knot

with the absolute value of the second coefficient of its Conway polynomial greater than or equal to n . In [2] m satisfies $m = 2p(15p - 9) \geq 8\sqrt{n}(60\sqrt{n} - 9) = 480n - 72\sqrt{n}$. In this paper we show that $m \geq 96\sqrt{n}$ is sufficient. Namely we show the following results.

Theorem 3. *Let k be a non-negative integer. Let $f : K_{6 \cdot 2^k} \rightarrow R^3$ be an embedding. Then there are mutually disjoint $3 \cdot 2^k$ -cycles γ_1 and γ_2 of $K_{6 \cdot 2^k}$ such that $|lk(f(\gamma_1), f(\gamma_2))| \geq 2^k$.*

Corollary 4. *Let n be a natural number. Let $f : K_{12n} \rightarrow R^3$ be an embedding. Then there are mutually disjoint cycles γ_1 and γ_2 of K_{12n} such that $|lk(f(\gamma_1), f(\gamma_2))| \geq n$.*

Theorem 5. *Let k be a non-negative integer. Let $f : K_{48 \cdot 2^k} \rightarrow R^3$ be an embedding. Then there is a cycle γ of $K_{48 \cdot 2^k}$ such that $|a_2(f(\gamma))| \geq 2^{2k}$.*

Corollary 6. *Let n be a natural number. Let m be a natural number with $m \geq 96\sqrt{n}$. Let $f : K_m \rightarrow R^3$ be an embedding. Then there is a cycle γ of K_m such that $|a_2(f(\gamma))| \geq n$.*

Remark. Let n be a natural number. Then we have shown that there is a natural number m such that every embedding of K_m contains a link whose absolute linking number is greater than or equal to n . The problem of deciding the smallest such m is still open. A similar problem for the second coefficient of the Conway polynomial of a knot is also still open. See [2].

Historical remark. The authors started this work without knowing Flapan's work [2]. The authors have shown Theorem 3 and Corollary 4 before knowing Flapan's work. Though the proof of Theorem 3 looks similar to that in [2], the proof is done independently. Then the authors have noticed Flapan's work. Then the authors have shown Theorem 5 and Corollary 6. The proof of Theorem 5 is essentially the same as that of Theorem 2 (2) in [2]. The proof of Theorem 2 (2) in [2] is based on Theorem 2 (1). The proof of Theorem 5 is based on Theorem 3. This is the only difference between them.

§2. Proofs

Lemma 7. *Let n and l be natural numbers with $2 \leq n < l$. Let G_l be a graph whose vertices are $v_1, v_2, \dots, v_l, u_1, u_2, \dots, u_l$ and whose edges are $v_1v_2, v_2v_3, \dots, v_{l-1}v_l, v_lv_1, u_1u_2, u_2u_3, \dots, u_{l-1}u_l, u_lu_1, v_1u_1, v_2u_2, \dots, v_lu_l$. Let α be the l -cycle containing v_1, v_2, \dots, v_l and β the l -cycle containing u_1, u_2, \dots, u_l . We give orientations to α and β by the cyclic orderings $v_1, v_2, \dots, v_{l-1}, v_l, v_1$ and $u_1, u_l, u_{l-1}, \dots, u_2, u_1$ respectively. Let $f : G_l \rightarrow R^3$ be an embedding. Let L be an oriented link in $R^3 - f(G_l)$. Suppose that the total linking number $lk(f(\alpha \cup \beta), L) \geq n$. Then there is an oriented $2l$ -cycle γ of G_l such that $lk(f(\gamma), L) \geq n$.*

Proof. Let c_i be the 4-cycle of G_l containing $v_i, u_i, v_{i+1}, u_{i+1}$ for $i = 1, 2, \dots, l-1$ and let c_l be the 4-cycle of G_l containing v_l, u_l, v_1, u_1 . We give orientations to these 4-cycles so that $\alpha + \beta + c_1 + c_2 + \dots + c_l = 0$ as elements of the first homology group of G . Then we have $lk(f(\alpha \cup \beta), L) + lk(f(c_1), L) + lk(f(c_2), L) + \dots + lk(f(c_l), L) = 0$. We will show that there exists $i \in \{1, 2, \dots, l\}$ such that $lk(f(c_i), L) \geq n - lk(f(\alpha \cup \beta), L)$. Suppose contrary that $lk(f(c_i), L) \leq n - lk(f(\alpha \cup \beta), L) - 1$ for each $i \in \{1, 2, \dots, l\}$. Then we have

$$\begin{aligned} 0 &= lk(f(\alpha \cup \beta), L) + lk(f(c_1), L) + lk(f(c_2), L) + \dots + lk(f(c_l), L) \\ &\leq lk(f(\alpha \cup \beta), L) + l(n - lk(f(\alpha \cup \beta), L) - 1) \\ &\leq lk(f(\alpha \cup \beta), L) + (n+1)(n - lk(f(\alpha \cup \beta), L) - 1) = n(n - lk(f(\alpha \cup \beta), L)) - 1 < 0. \end{aligned}$$

This is a contradiction. Thus we have that there exists $i \in \{1, 2, \dots, l\}$ such that $lk(f(c_i), L) \geq n - lk(f(\alpha \cup \beta), L)$. Let $\gamma = \alpha + \beta + c_i$. Then γ is an oriented $2l$ -cycle such that $lk(f(\gamma), L) = lk(f(\alpha \cup \beta), L) + lk(f(c_i), L) \geq n$. This completes the proof. \square

Proof of Theorem 3. The proof is given by an induction on k . The case $k = 0$ is just Theorem 1 (1). Suppose that it is shown for k . Let $f : K_{6 \cdot 2^{k+1}} \rightarrow R^3$ be an embedding. Let H_1, H_2 be mutually disjoint subgraphs of $K_{6 \cdot 2^{k+1}}$ each of which is isomorphic to $K_{6 \cdot 2^k}$. Then by the assumption there are mutually disjoint $3 \cdot 2^k$ -cycles $\beta_1, \beta_2, \beta_3, \beta_4$ with $\beta_1 \cup \beta_2 \subset H_1$ and $\beta_3 \cup \beta_4 \subset H_2$ such that $|lk(f(\beta_1), f(\beta_2))| \geq 2^k$ and $|lk(f(\beta_3), f(\beta_4))| \geq 2^k$. Then it is easy to see that we can choose orientations of these cycles so that $lk(f(\beta_1 \cup$

$\beta_3), f(\beta_2 \cup \beta_4)) \geq 2^{k+1}$.

Then we take a subgraph H of $K_{6 \cdot 2^{k+1}}$ that is isomorphic to $G_{3 \cdot 2^k}$ in Lemma 7 so that β_1 and β_3 correspond to α and β in Lemma 7. We set $L = f(\beta_2 \cup \beta_4)$ and apply Lemma 7. Then we have an oriented $3 \cdot 2^{k+1}$ -cycle $\gamma \subset H$ such that $lk(f(\gamma), L) \geq 2^{k+1}$.

Next we set $L' = f(\gamma)$ and take a subgraph H' of $K_{6 \cdot 2^{k+1}}$ that is isomorphic to $G_{3 \cdot 2^k}$ in Lemma 7 so that β_2 and β_4 correspond to α and β in Lemma 7. Then we apply Lemma 7 again and have an oriented $3 \cdot 2^{k+1}$ -cycle $\gamma' \subset H'$ such that $lk(f(\gamma), f(\gamma')) \geq 2^{k+1}$. This completes the proof. \square

Proof of Corollary 4. Let n be a natural number and $f : K_{12n} \rightarrow R^3$ an embedding. Then there is an integer k such that $2^{k-1} < n \leq 2^k$. Then we have $2^k < 2n$ and therefore $6 \cdot 2^k < 12n$. Therefore K_{12n} contains a subgraph that is isomorphic to $K_{6 \cdot 2^k}$. Therefore by Theorem 3 we have that there are mutually disjoint cycles γ_1 and γ_2 of K_{12n} such that $|lk(f(\gamma_1), f(\gamma_2))| \geq 2^k \geq n$. This completes the proof. \square

Let G_4 and G_8 be graphs as illustrated in Fig. 1.

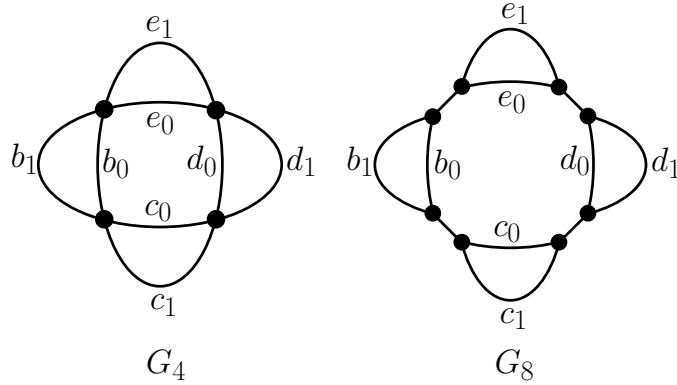


Fig. 1

Let $\Gamma(G_4)$ and $\Gamma(G_8)$ be the set of all 4-cycles of G_4 and the set of all 8-cycles of G_8 respectively. Let $i, j, k, l \in \{0, 1\}$. By $b_i c_j d_k e_l$ we denote the 4-cycle of G_4 or the 8-cycle of G_8 that contains the edges b_i, c_j, d_k and e_l . For each such cycle we define

$$\varepsilon(b_i c_j d_k e_l) = \begin{cases} 1 & \text{if } i + j + k + l \text{ is even,} \\ -1 & \text{if } i + j + k + l \text{ is odd.} \end{cases}$$

Let $f : G_4 \rightarrow R^3$ be an embedding. Let $\lambda(f)$ be an integer defined by

$$\lambda(f) = \left| \sum_{\gamma \in \Gamma(G_4)} \varepsilon(\gamma) a_2(f(\gamma)) \right|.$$

Let $f : G_8 \rightarrow R^3$ be an embedding. Let $\lambda(f)$ be an integer defined by

$$\lambda(f) = \left| \sum_{\gamma \in \Gamma(G_8)} \varepsilon(\gamma) a_2(f(\gamma)) \right|.$$

Let $B, C, D,$ and E denote the 2-cycles $b_0 \cup b_1, c_0 \cup c_1, d_0 \cup d_1,$ and $e_0 \cup e_1$ respectively. It is shown in [3] that for any embedding $f : G_4 \rightarrow R^3$ we have the equation

$$\lambda(f) = |lk(f(B), f(D))lk(f(C), f(E))|.$$

By contracting the four edges of G_8 we also have the same equation for any embedding $f : G_8 \rightarrow R^3$. Since $\Gamma(G_8)$ contains 16 cycles we have that

$$16 \max\{|a_2(f(\gamma))| \mid \gamma \in \Gamma(G_8)\} \geq \lambda(f) = |lk(f(B), f(D))lk(f(C), f(E))|.$$

Therefore we have that

$$\max\{|a_2(f(\gamma))| \mid \gamma \in \Gamma(G_8)\} \geq \frac{1}{16} |lk(f(B), f(D))lk(f(C), f(E))|.$$

Proof of Theorem 5. Let H_1 and H_2 be mutually disjoint subgraphs of $K_{48 \cdot 2^k}$ each of which is isomorphic to $K_{24 \cdot 2^k} = K_{6 \cdot 2^{k+2}}$. Let $f : K_{48 \cdot 2^k} \rightarrow R^3$ be an embedding. Then by Theorem 3 there are mutually disjoint cycles γ_1 and γ_2 of H_1 and mutually disjoint cycles γ_3 and γ_4 of H_2 such that $|lk(f(\gamma_1), f(\gamma_2))| \geq 2^{k+2}$ and $|lk(f(\gamma_3), f(\gamma_4))| \geq 2^{k+2}$. Let H_3 be a subgraph of $K_{48 \cdot 2^k}$ that is isomorphic to G_8 that contains $\gamma_1, \gamma_3, \gamma_2$ and γ_4 in this cyclic order. Then we have that there is a cycle γ of H_3 such that $|a_2(f(\gamma))| \geq \frac{1}{16} 2^{k+2} \cdot 2^{k+2} = 2^{2k}$. This completes the proof. \square

Proof of Corollary 6. Let n be a natural number. Let m be a natural number with $m \geq 96\sqrt{n}$ and let $f : K_m \rightarrow R^3$ be an embedding. Then there is a non-negative

integer k such that $2^{2(k-1)} < n \leq 2^{2k}$. Then we have $2^{k-1} < \sqrt{n}$. Therefore we have $48 \cdot 2^k = 96 \cdot 2^{k-1} < 96\sqrt{n}$. Therefore K_m contains a subgraph isomorphic to $K_{48 \cdot 2^k}$. Then by Theorem 5 we have that there is a cycle γ of K_m such that $|a_2(f(\gamma))| \geq 2^{2k} \geq n$. This completes the proof. \square

References

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